COMMUTATIVE DIAGRAMS

Definition 1. A *diagram* in a category $\mathcal{C}$ comprises a graph $\mathcal{J}$ (the *shape*) and a graph homomorphism $D\colon \mathcal{J} \to \mathcal{C}$ from $\mathcal{J}$ to the underlying graph of $\mathcal{C}$.

Definition 2. A diagram $(\mathcal{J}, D)$ in $\mathcal{C}$ is *commutative* if, for every pair of paths in $\mathcal{J}$ (i.e. sequence of edges)

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} x_{k-1} \xrightarrow{f_k} x_k$$

and

$$y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2 \xrightarrow{g_3} \cdots \xrightarrow{g_{\ell-1}} y_{\ell-1} \xrightarrow{g_\ell} y_\ell$$

where $x_0 = y_0$ and $x_k = y_\ell$ (i.e. the paths start at the same node, and end at the same node), we have

$$D(f_k) \circ D(f_{k-1}) \circ \cdots \circ D(f_3) \circ D(f_2) \circ D(f_1) = D(g_\ell) \circ D(g_{\ell-1}) \circ \cdots \circ D(g_3) \circ D(g_2) \circ D(g_1)$$

in $\mathcal{C}$. (Above, the two paths must have length at least 1 (i.e. $1 \leq k$ and $1 \leq \ell$), but the two paths do not have to have the same length, e.g. k may not be equal to ℓ.)

Example 3. Let $\mathcal{J}$ be the following graph.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

A diagram with shape $\mathcal{J}$ in a category $\mathcal{C}$ picks out three (not necessarily different) objects $D(A), D(B), D(C) \in \mathcal{C}_0$, and morphisms $D(f)\colon D(A) \to D(B)$, $D(g)\colon D(B) \to D(C)$, $D(h)\colon D(A) \to D(C)$.

$$\begin{tikzcd} D(A) \arrow[r, "D(f)"] \arrow[dr, "D(h)"'] & D(B) \arrow[d, "D(g)"] \\ & D(C) \end{tikzcd}$$

Such a diagram commutes if and only if the we have

$$D(g) \circ D(f) = D(h)$$

in $\mathcal{C}$. For instance, the following diagram commutes in the category of sets and total functions (where $D(A) = \mathbb{N}, D(B) = \mathbb{N}, D(C) = \mathbb{N}$ and $f = (-) + 1$, $g = (-) + 1$, $h = (-) + 2$).

$$\begin{tikzcd} \mathbb{N} \arrow[r, "(-)+1"] \arrow[dr, "(-)+2"'] & \mathbb{N} \arrow[d, "(-)+1"] \\ & \mathbb{N} \end{tikzcd}$$

Date: 22 February 2024.