Definition 1. A *diagram* in a category C comprises a graph \mathcal{J} (the *shape*) and a graph homomorphism $D: \mathcal{J} \to C$ from \mathcal{J} to the underlying graph of C.

Definition 2. A diagram (\mathcal{J}, D) in \mathcal{C} is *commutative* if, for every pair of paths in \mathcal{J} (i.e. sequence of edges)

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{k-1}} x_{k-1} \xrightarrow{f_k} x_k$$

and

$$y_0 \xrightarrow{g_1} y_1 \xrightarrow{g_2} y_2 \xrightarrow{g_3} \cdots \xrightarrow{g_{\ell-1}} y_{\ell-1} \xrightarrow{g_\ell} y_\ell$$

where $x_0 = y_0$ and $x_k = y_\ell$ (i.e. the paths start at the same node, and end at the same node), we have

$$D(f_k) \circ D(f_{k-1}) \circ \cdots \circ D(f_3) \circ D(f_2) \circ D(f_1) = D(g_\ell) \circ D(g_{\ell-1}) \circ \cdots \circ D(g_3) \circ D(g_2) \circ D(g_1)$$

in C. (Above, the two paths must have length at least 1 (i.e. $1 \le k$ and $1 \le \ell$), but the two paths do not have to have the same length, e.g. k may not be equal to ℓ .)

Example 3. Let \mathcal{J} be the following graph.



A diagram with shape \mathcal{J} in a category \mathcal{C} picks out three (not necessarily different) objects $D(A), D(B), D(C) \in \mathcal{C}_0$, and morphisms $D(f): D(A) \to D(B), D(g): D(B) \to D(C), D(h): D(A) \to D(C).$

$$D(A) \xrightarrow{D(f)} D(B)$$

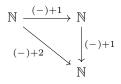
$$\downarrow D(g)$$

$$D(C)$$

Such a diagram commutes if and only if the we have

$$D(g) \circ D(f) = D(h)$$

in C. For instance, the following diagram commutes in the category of sets and total functions (where $D(A) = \mathbb{N}, D(B) = \mathbb{N}, D(C) = \mathbb{N}$ and f = (-) + 1, g = (-) + 1, h = (-) + 2).



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