Reference Cards

Monoidal and enriched derivators

Thursday 21st September, 2017 14:16

A nice feature of rings is that they behave like monoidal categories with one object (or vice versa).

- Any monoidal functor $F : \mathcal{V} \to \mathcal{W}$ (lax is enough) induces a base change 2-functor
$$F_* : \mathcal{V}\text{-Cat} \longrightarrow \mathcal{W}\text{-Cat}$$
that sends a $\mathcal{V}$-category $\mathcal{C}$ into the $\mathcal{W}$-category having the same objects of $\mathcal{C}$ and where $(F_*\mathcal{C})(X, Y) = F(\mathcal{C}(X, Y))$.
- The structural 2-cells of F induce the monoidal structure on $F_*\mathcal{C}$.
- Monoidal transformations are induced accordingly (the definition is straightforward): a natural transformation $\beta : F \to G$ induces a 2-natural transformation between the 2-functors F_* and G_* with 'restricted' components.

It seems that this construction could be applied to $\mathcal{V} \to \mathbf{Set}$ to generate the underlying functor $U : \mathcal{V}\text{-Cat} \to \text{Cat}$, but the fact is that $\text{hom}(J, -)$ is seldom monoidal.

The assignment described above that sends $\mathcal{V}$ into $\mathcal{V}$-Cat, F to F_* and β to β_* is a 2-functor

$$(-)\text{-Cat} : \text{Cat}_{\otimes} \longrightarrow 2\text{-Cat}$$

A suitable 2-categorical Grothendieck construction gives rise then to a universal fibration

$$\begin{array}{c} \text{EnCat} \\ \big\downarrow p \\ \text{Cat}_{\otimes} \end{array}$$

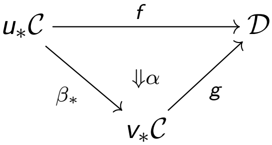
whose fiber over $\mathcal{V}$ is the 2-category of $\mathcal{V}$-categories.

- This is no different from the construction of the fibration $\textbf{Mod} \to \textbf{Ring}$ whose fiber over the ring R is the category R-Mod of modules over R. This is the canonical fibration for $F : \textbf{Ring} \to \textbf{Cat}$, and $\textbf{Mod} = \int_1 F$.
- General definitions pertaining the Grothendieck construction apply here and we have a definition on functors and natural transformations.

- A morphism $(\mathcal{V}, \mathcal{C}) \to (\mathcal{W}, \mathcal{D})$ in EnCat is given by a pair $u : \mathcal{V} \to \mathcal{W}$ and a functor $f : u_*\mathcal{C} \to \mathcal{D}$. Composition is given by

$$(vu)_*\mathcal{C} = v_* u_* \mathcal{C} \xrightarrow{v_* f} v_* \mathcal{D} \xrightarrow{g} \mathcal{E}$$

- A 2-cell $\alpha : (u, f) \to (v, g)$ is defined for two parallel 1-cells $(\mathcal{V}, \mathcal{C}) \to (\mathcal{W}, \mathcal{D})$ as a pair $\beta : u \to v$ (which is monoidal) and α is a 2-cell



All the forgetful functors $U_{\mathcal{V}} : \mathcal{V}\text{-Cat} \to \text{Cat}$ glue together to form a functor

$$U : \text{EnCat} \to \text{Cat}$$

defined by $U(\mathcal{V}, \mathcal{C}) = U_{\mathcal{V}}(\mathcal{C})$ =the underlying unenriched category of $\mathcal{C}$. All the compatibility check are straightforward.

Recall that a monoidal prederivator is a strict 2-functor $\mathbb{E} : \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Cat}_{\otimes}$. A prederivator enriched over $\mathbb{E}$ is a 2-functor $\mathbb{D}$ such that $p \circ \mathbb{D} = \mathbb{E}$.

The essential of this definition is: an enriched derivator specifies an $\mathbb{E}(J)$-enriched category $\mathbb{D}(J)$ for each $J \in \mathrm{Cat}$, and this specification is 2-functorial in J. Graphically,

$$\begin{array}{ccccc} \mathrm{Cat}^{\mathrm{op}} & \xrightarrow{\mathbb{D}} & \mathrm{EnCat} & \xrightarrow{U} & \mathrm{CAT} \\ & \searrow^{\mathbb{E}} & \downarrow p & & \\ & & \mathrm{Cat}_{\otimes} & & \end{array}$$

The composition $U \circ \mathbb{D}$ is the prederivator underlying the enriched prederivator $\mathbb{D}$.

Defining a morphism of enriched prederivators is notationally quite painful, but the definition is clear: it's a pseudonatural transformation between 2-functors $\mathsf{Cat}^{\mathsf{op}} \to \mathsf{EnCat}$.

From the definition of morphism in EnCat it follows that we have to specify a pseudonatural transformation $F : \mathbb{D} \to \mathbb{D}'$ whose components $F_I : \mathbb{D}(I) \to \mathbb{D}'(I)$ satisfy the commutativity

$$\begin{array}{ccc}
\mathbb{E}(u)_*\mathbb{D}(K) & \longrightarrow & \mathbb{E}(u)_*\mathbb{D}'(K) \\
\downarrow & \swarrow & \downarrow \\
\mathbb{D}(J) & \xrightarrow[F_J]{} & \mathbb{D}'(J)
\end{array}$$

for each $u : J \to K$, where we exceptionally denoted $\mathbb{E}(u)$ the action of $\mathbb{E}$ on u.
(the yoga is: as a monoidal functor, $\mathbb{E}(u)$ turns $\mathbb{D}(K)$ into a $\mathbb{E}(J)$-enriched category, and then the square above is the only way to compare them according to the def. of morphisms in EnCat).

A general result in enriched stuff is:

Theorem

Given a 2v adjunction $\mathcal{E} \times \mathcal{D} \overset{\curvearrowright}{\longrightarrow} \mathcal{D}$ where $\mathcal{E}$ is monoidal and $\mathcal{D}$ is $\mathcal{E}$-tensored. Then $\mathcal{D}$ is also $\mathcal{E}$-cotensored and canonically $\mathcal{E}$-enriched.

We want to show that this is the base case of a theorem on derivators:

Theorem for derivators

Let $\mathbb{E}$ be a monoidal derivator, and *bD* tensored over $\mathbb{E}$. If there is a 2v adjunction inducing the tensoring,

$$(\otimes, \mathrm{hom}_l, \mathrm{hom}_r) : \mathbb{E} \times \mathbb{D} \to \mathbb{D}$$

then $\mathbb{D}$ is canonically $\mathbb{E}$-enriched and cotensored.

From the definition of an 2v adjunction for derivators we get that each $\mathbb{D}(K)$ is tensored over $\mathbb{E}(K)$ and part of a 2v adjunction

$$(\otimes, \mathrm{HOM}_{l,\mathbb{D}(K)}, \mathrm{HOM}_{r,\mathbb{D}(K)}) : \mathbb{E}(K) \times \mathbb{D}(K) \to \mathbb{D}(K)$$

Using the result for plain categories we get that each $\mathbb{D}(K)$ is enriched over $\mathbb{E}(K)$, and we prove that it is coherently so: hom_r will give all the needed coherence.

As a general tenet, if you can do something in model categories you can do it in derivators:

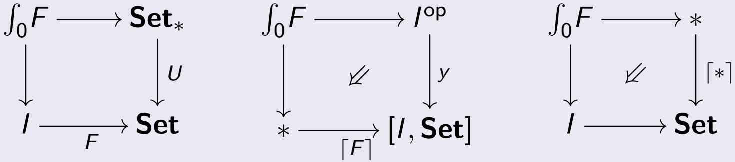
If $\mathcal{M}, \mathcal{N}$ are combinatorial model categories, $\mathcal{M}$ is also monoidal, and $\mathcal{N}$ is $\mathcal{M}$-tensored, then the derivator $\mathbb{D}_{\mathcal{N}}$ is canonically tensored, cotensored and enriched over the monoidal derivator $\mathbb{D}_{\mathcal{M}}$.

This applies to **sSet**-model categories, **Sp**-model categories, $\mathbf{dg}_k$-model categories. . .

The Grothendieck construction

The previous construction of p makes heavy use of the Grothendieck construction for 2-categories. We recall it starting from its 0-dimensional counterpart.

For a functor $F : I \to \mathbf{Set}$ all you need to know is in any of these equivalent universal properties:



There is a fibration $p : \int_0 F \to I$ such that $p^{-1} i$ is the set $F(i)$.

For a functor $F : I \to \mathbf{Cat}$ we define $\int_1 F$ as the category of pairs $(i, X \in F(i))$, and a morphism $(i, X) \to (j, Y)$ to be a pair (f, u) such that $f : i \to j$ and $u : F(f)X \to Y$ in $F(j)$. Composition is defined as

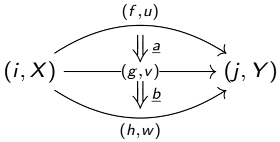
$$(i, X) \stackrel{(f,u)}{\to} (j, Y) \stackrel{(g,v)}{\to} (k, Z)$$
$$(i, X) \stackrel{(g.f, v.F(g)u)}{\to} (k, Z)$$

Again there is a fibration $p : \int_1 F \to I$ such that $p^{-1} i$ is a category isomorphic to $F(i)$.

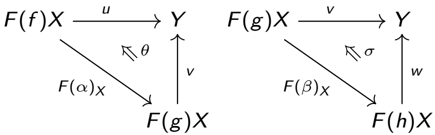
For a 2-functor $F : I \to 2\text{-}\mathbf{Cat}$, things go as expected but the definition is quite daunting: $\int F$ has $\int_1 F$ as underlying 1-category (in a similar manner, $\int_1 F$ had $\int_0 F$ as set of objects); 2-cells and their two compositions (horizontal and vertical) are defined as follows

- A 2-cell $(i, X) \underset{(g,v)}{\overset{(f,u)}{\Downarrow \underline{a}}} (j, Y)$ is a pair (α, θ) such that $\alpha : f \to g$ is a 2-cell in I and $\theta : v.F(\alpha)_X \to u$ is a 2-cell in $F(j)$.

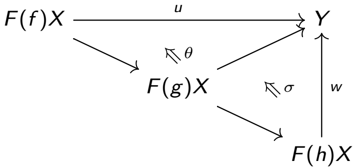
- Horizontal composition is defined for cells



i.e. for diagrams of 2-cells like

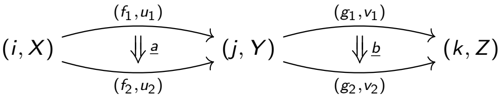


as the pasting

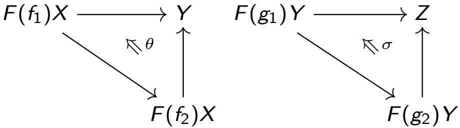


giving that $(\beta, \sigma) \circ_v (\alpha, \theta) = (\beta \circ_v \alpha, (\sigma * F(\alpha)_X) \circ \theta)$.

- Vertical composition is defined for cells



i.e. for diagrams of 2-cells like



as the pasting