TALLINN UNIVERSITY OF TECHNOLOGY

# Category Theory and its Applications – $\rm ITI9200$

Exercise sheet 1

assigned: March 12, 2025 due: March 26, 2025 An intuitive representation of the slice category  $\mathbf{Set}/A$  is the following. Suppose  $\{1, \ldots, r\}$  is a set with r elements; an r-colouring of a set X can be seens as a function  $c: X \to \{1, \ldots, r\}$  (so that the elements of  $X_i := c^{-1}(i) := \{x \in X \mid c(x) = i\}$  are the elements 'coloured' by i, if i is understood as a 'colour'  $1 \le i \le r$ ); many problems in elementary enumerative combinatorics concern colourings of finite sets, and thus they can be stated as problems about the slice category  $\mathbf{Set}/\{1, \ldots, r\}$ .



This exercises is meant to let you study *coloured graphs*. A (undirected, loop-free) graph  $\mathcal{G}$  is made of a set of vertices V and a set of edges E, which can be thought as a set of 2-element subsets of V. If  $\{v_1, v_2\} \in E$ , we say that  $v_1, v_2$  are *adjacent* in the graph  $\mathcal{G}$ .

#### Exercise 1:

Define an *r*-colouring of an undirected, loop-free graph  $\mathcal{G} = (E, V)$  as a function  $c: V \to \{1, \ldots, r\}$ , which assigns a colour  $1 \leq j \leq r$  to each vertex V of  $\mathcal{G}$ , in such a way that two vertices  $v_1, v_2$  connected by an edge are not of the same colour. Homomorphisms of coloured graphs *preserve the r-colouring* of domain and codomain, that is, they map vertices of one colour to vertices of the same colour.

Let  $\mathcal{K}_r$  be the graph defined as follows (it's the *complete graph* on r vertices):

- the vertices of  $\mathcal{K}_r$  are the elements of  $\{1, \ldots, r\}$ ;
- two vertices i, j are adjacent if and only if  $i \neq j$ .

▶ Prove that an *r*-colouring of a graph  $\mathcal{G}$  corresponds precisely to the data of a graph homomorphism  $\mathcal{G} \to \mathcal{K}_r$ .

▶ ■ Deduce that the category of coloured graphs is identified with the slice category  $\mathbf{Gph}/\mathcal{K}_r$ .

A binary operation of fundamental importance in the theory of categories is the *join* of categories.

## Exercise 2:

Given two categories C, D, the *join* of C and D, denoted by  $C \star D$ , is the category defined as follows:

- the objects are the disjoint union of  $C_0$  and  $\mathcal{D}_0$  (that is, the same objects as the sum  $\mathcal{C} + \mathcal{D}$ );
- the set of morphisms  $(\mathcal{C} \star \mathcal{D})(X, Y)$  are specified by case splitting:
  - if X = C, Y = C' are both in  $\mathcal{C}_0$ , then  $(\mathcal{C} \star \mathcal{D})(X, Y) := \mathcal{C}(C, C')$ ;
  - if X = D, Y = D' are both in  $\mathcal{D}_0$ , then  $(\mathcal{C} \star \mathcal{D})(X, Y) := \mathcal{D}(D, D')$ ;
  - if  $X = C \in \mathcal{C}_0$  and  $Y = D \in \mathcal{D}_0$ , there exists a *unique* arrow  $u_{CD} : C \rightsquigarrow D$ ; such  $u_{CD}$  is called the *heteromorphism* connecting C and D;
  - if  $X = D \in \mathcal{D}_0$  and  $Y = C \in \mathcal{C}_0$ , then  $(\mathcal{C} \star \mathcal{D})(X, Y)$  is empty.

Composition is defined as in C (or as in D) when both morphisms are in C (or in D); composition with a heteromorphism always results in a heteromorphism:

$$\begin{array}{rcl} \forall f: C & \stackrel{f}{\longrightarrow} C' & \stackrel{u_{C'D}}{\dashrightarrow} D & = & C & \stackrel{u_{CD}}{\dashrightarrow} D \\ \forall g: C & \stackrel{u_{CD}}{\dashrightarrow} D & \stackrel{g}{\longrightarrow} D' & = & C & \stackrel{u_{CD}}{$ \stackrel{u_{CD'}}{$ \xrightarrow{} D' $} } D' \end{array}$$

▶ Show by induction that if  $n \ge 2$  the composition of an *n*-tuple of morphisms in  $\mathcal{C} \star \mathcal{D}$ , of which at least one is a heteromorphism, is a heteromorphism (between the appropriate domain and codomain). Then prove that the join  $\mathcal{C} \star \mathcal{D}$  is indeed a category.

▶ Determine the join of two chains  $\Delta[n] \star \Delta[m]$ , if  $\Delta[n]$  is the category  $\{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$ .

▶ ■ Determine the join of two cubes  $P[n] \star P[m]$ , if P[n] is the *n*-dimensional cube of subsets of  $\{1, \ldots, n\}$ ; more generally, how can one represent the join of two partially ordered sets P, Q in terms of P, Q?

▶ ■ Determine the join of two discrete finite categories  $A^{\delta} \star B^{\delta}$ , and of two codiscrete categories  $A^{\chi} \star B^{\chi}$ : is the result still discrete, still codiscrete?

▶ Determine the iterated join  $\mathcal{B}(\mathbb{N}, +, 0) \star \mathcal{B}(\mathbb{N}, +, 0)$  of the additive monoid of natural numbers, and the iterated join  $\mathcal{B}(\mathbb{N}, \max, 0) \star \mathcal{B}(\mathbb{N}, \max, 0)$  of the monoid under the maximum operation,  $n \vee m := \max\{n, m\};^{a}$ 

<sup>&</sup>lt;sup>a</sup>If M is a monoid, to avoid ambiguity, we denote by  $\mathcal{B}M$  the category with a single object defined by M in the usual way.

▶ ■ Determine whether there is a relation between  $(C \star D)^{\text{op}}$  and  $C^{\text{op}} \star D^{\text{op}}$  (equal, opposite to each other, no relation...)?

Recall the construction of the right cone  $S^{\triangleright}$  over a set S: it is the category whose objects are  $S \cup \{\infty\}$  and whose arrows are the set  $\{\lambda_s : s \to \infty \mid s \in S\}$ . Similarly,  $S^{\triangleleft}$  is the category whose objects are  $S \cup \{-\infty\}$  and whose arrows are the set  $\{\omega_s : -\infty \to s \mid s \in S\}$ .

▶ Express  $S^{\triangleright}$  and  $S^{\triangleleft}$  as joins of two categories; describe  $S^{\triangleleft} \star S^{\triangleright}$ ,  $S^{\triangleleft} \star S^{\triangleleft}$ ,  $S^{\triangleright} \star S^{\triangleright}$ ,  $S^{\triangleright} \star S^{\triangleleft}$ .

Combine the constructions of product, sum, slice, and coslice with the join construction:

► Can the join  $(\mathcal{A} \times \mathcal{B}) \star \mathcal{C}$  be described in terms of  $\mathcal{A} \star \mathcal{C}$ ,  $\mathcal{B} \star \mathcal{C}$ ? How can  $(\mathcal{C} \star \mathcal{D})/X$  be described as X varies in  $\mathcal{C}_0 + \mathcal{D}_0$ ? How can  $(\mathcal{A}/A) \star (\mathcal{B}/\mathcal{B})$ ,  $(\mathcal{A}/\mathcal{A}) \star (\mathcal{B}/\mathcal{B})$ ,  $(\mathcal{A}/\mathcal{A}) \star (\mathcal{B}/\mathcal{B})$ ,  $(\mathcal{A}/\mathcal{A}) \star (\mathcal{B}/\mathcal{B})$ , be described in terms of simpler components? How can  $(\mathcal{A} \star \mathcal{B}) + \mathcal{C}$  be described in terms of simpler components?

This exercise plays with three different representations of the category **Rel** of (finite) sets and relations.

Exercise 3:

Three versions of  $\mathbf{Rel}$  (link).

**Relations as spans.** A relation between finite sets A, B consists of a subset R of the Cartesian product  $A \times B$ . As such, a relation  $R \subseteq A \times B$  corresponds to a span

$$A \xleftarrow{a} R \xrightarrow{b} B$$

**Relations as cospans, or rather, as graphs.** The cospan c(R) associated with the relation R is defined as the category whose objects are the elements of the disjoint union of sets A + B, and where there is an arrow  $s \to t$  if and only if  $s \in A, t \in B$ , and  $(s,t) \in R$ . This cospan

$$A \longrightarrow c(R) \longleftarrow B$$

is called the graph of R: to understand why,

▶ Draw the cospan c(R) for simple relations between finite sets; fix a set X and characterize the properties of c(R) for a relation  $R \subseteq X \times X$  that is reflexive, symmetric, and transitive on X.

▶ ■ What is the category c(f) associated with a function, viewed as a total and single-valued relation?<sup>*a*</sup>

<sup>a</sup>Recall that a 'function'  $f \subseteq A \times B$  is a relation f that satisfies the following property: for each  $a \in A$ , there exists a unique  $b \in B$  such that  $(a, b) \in f$ .

▶ ■ Given R as subset, what is *exactly* the relation between the span with tip R and the cospan with bottom c(R)? Are they interdefinable? Does one contain strictly more information than the other?

**Relations as matrices.** A relation  $R \subseteq X \times Y$  can also be represented as a matrix whose entries are Boolean values: if  $X = \{x_1, \ldots, x_n\}$  and  $Y = \{y_1, \ldots, y_m\}$ , then R consists of the matrix  $(r_{ij})$  with entries 0 or 1, so that  $r_{ij} = 1$  if and only if  $(x_i, y_j) \in R$ , and zero otherwise.

For the sake of concreteness, from now on let's just consider the special finite sets  $[n] := \{1, \ldots, n\}$  as objects.

For example, if n = 4, m = 7, the matrix

$$R = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

represents a relation  $R : \{1, \ldots, 7\} \rightarrow \{1, \ldots, 4\}$ : write it as a subset of  $\{1, \ldots, 7\} \times \{1, \ldots, 4\} = \{(i, j) \mid 1 \le i \le 7, 1 \le j \le 4\}$ ; then depict its graph c(R).

$$(S \circ R)_{ij} = \bigvee_{k=1}^{n} r_{ik} \wedge s_{kj}.^{a}$$

<sup>a</sup>Recall that  $\wedge$  and  $\vee$  are the usual Boolean operations of logical AND and OR:

$\wedge$	0	1	V	0	1
0	0	0	0	0	1
1	0	1	1	1	1

*Pointwise operations* between Boolean matrices are induced by said operations of logical AND, OR and NOT between Booleans; they are defined as follows:

- matrix OR: given  $R, S : A \to B$  the matrix  $R \lor S$  is defined as  $(R \lor S)_{ij} = R_{ij} \lor S_{ij}$ ;
- matrix AND: given  $R, S : A \to B$  the matrix  $R \land S$  is defined as  $(R \land S)_{ij} = R_{ij} \land S_{ij}$ ;
- matrix NOT: given  $R: A \to B$ , the matrix  $\neg R$  is defined as  $(\neg R)_{ij} = 0$  if and only if  $R_{ij} = 1$ ,  $(\neg R)_{ij} = 1$  if and only if  $R_{ij} = 0$ .

 $\blacktriangleright$   $\square$  Define all these operations as operations on relations-as-spans and relations-as-graphs.

▶ ■ Are the following properties true for all Boolean matrices R, S, T (of course, taking them of the appropriate size when needed)? (Be mindful of what is true in linear algebra and what is false!)

- $(R \lor S) \land T = (R \land T) \lor (S \land T)$ , and  $R \lor (S \land T) = (R \lor S) \land (R \lor T)$ .
- $(R \land S) \circ T = (R \circ T) \land (S \circ T)$ , and  $R \circ (S \lor T) = (R \circ S) \lor (R \circ T)$ .
- $\neg(R \land S) = (\neg R) \lor (\neg S)$  and  $\neg(R \lor S) = (\neg R) \land (\neg S)$ .
- $(\neg R) \circ S = R \circ (\neg S).$

A form of *Kronecker product* of Boolean matrices is also defined: given  $R: A \to B$ and  $S: X \to Y$  two Boolean matrices (note that the sizes of domain and codomain of  $R = (r_{ij}), S = (s_{pq})$  are possibly not related in any way) the Boolean matrix  $R \otimes S$ :  $A \times X \to B \times Y$  is defined as follows:

$$R \otimes S = \begin{pmatrix} r_{11}S & \cdots & r_{1n}S \\ \vdots & \ddots & \vdots \\ r_{m1}S & \cdots & r_{mn}S \end{pmatrix} = \begin{pmatrix} r_{11}s_{11} & r_{11}s_{12} & \cdots & r_{11}s_{1q} & \cdots & \cdots & r_{1n}s_{11} & r_{1n}s_{12} & \cdots & r_{1n}s_{1q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{11}s_{p1} & r_{11}s_{p2} & \cdots & r_{11}s_{pq} & \cdots & \cdots & r_{1n}s_{p1} & r_{1n}s_{p2} & \cdots & r_{1n}s_{pq} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{11}s_{p1} & r_{11}s_{p2} & \cdots & r_{11}s_{pq} & \cdots & \cdots & r_{1n}s_{p1} & r_{1n}s_{p2} & \cdots & r_{1n}s_{pq} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ r_{m1}s_{11} & r_{m1}s_{12} & \cdots & r_{m1}s_{1q} & \cdots & \cdots & r_{mn}s_{11} & r_{mn}s_{12} & \cdots & r_{mn}s_{1q} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ r_{m1}s_{p1} & r_{m1}s_{p2} & \cdots & r_{m1}s_{pq} & \cdots & \cdots & r_{mn}s_{p1} & r_{mn}s_{p2} & \cdots & r_{mn}s_{pq} \end{pmatrix}$$

▶ Define Kronecker product in terms of relations-as-spans and relations-asgraphs.

▶ ■ Are the following identities true for all Boolean matrices R, S, T (once again, taking them of the appropriate size when needed)?

- $(R \land S) \otimes T = (R \otimes T) \land (S \otimes T)$ , and  $(R \lor S) \otimes T = (R \otimes T) \lor (S \otimes T)$ .
- $(R \circ S) \otimes (U \circ T) = (R \otimes U) \circ (S \otimes T)$ , and  $R \circ (S \otimes T) = (R \circ S) \otimes (R \circ T)$ .

• 
$$\neg (R \otimes S) = (\neg R) \otimes (\neg S).$$

Consider the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

▶ Compute  $A \land B$ ,  $A \lor B$ ,  $\neg A$ ; compute the powers  $P \circ P, P \circ P \circ P, \ldots, P^{\circ n}$  of P; compute  $Q \circ P$ ; compute  $B \otimes Q$  and compare it with  $Q \otimes B$ ; draw their associated relations-as-graphs.

Given all this, how distant is the category of relations-as-Boolean-matrices **MRel** from the category  $\mathbb{Z}/2\mathbb{Z}$ -**Mat** having objects the natural numbers (i.e. the finite sets  $[n] := \{1, \ldots, n\}$ ) and morphisms  $[n] \to [m]$  the  $m \times n$  matrices with entries in the field with two elements?

 $\blacktriangleright$   $\square$  Can one define functors

 $\mathbf{MRel} \longrightarrow \mathbb{Z}/2\mathbb{Z}\text{-}\mathbf{Mat} \qquad \mathbb{Z}/2\mathbb{Z}\text{-}\mathbf{Mat} \longrightarrow \mathbf{MRel}$ 

inverse to each other? Is every R 'linear'? How does one interpret the 'determinant' of a Boolean matrix  $R: n \to n$ ? Etc.

#### Exercise 4:

On the distant planet Kobaïa, centuries ago, the inhabitants faced a challenge: how to effectively navigate the inherent ambiguity of Kobaïan language? After an era of profound intellectual curiosity, followed by three disastrous planetary wars, the inhabitants turned to the only possible source of objectivity and peace —category theory.

After the reform operated by Emperor Horžtavak, Kobaïa contains three artificial languages: *poetic*, *structural*, and *imperial*.

- **Poetic** Kobaïan contains nouns, adjectives, adverbs and verbs. It is willingly kept as morphologically simple as possible: verbs have no conjugations, there is no marker to distinguish subject and object, there is no distinction of gender, no plural form... In short, poetic Kobaïan is (approximated by what on planet Earth we call) a *closed, isolating language.*
- Structural Kobaïan opts for an intermediate solution, as it is used for objective, everyday communication; there is a difference between an animate and inanimate object, a prefix ( $w\ddot{o}$ -) transforms nouns into adjectives<sup>1</sup> and a prefix  $\hat{a}$  to turn adjectives into adverbs,<sup>2</sup> there is a system of pronouns<sup>3</sup>
- Imperial Kobaïan is reserved for communication with the *Lusztzess* (the Emperor) and the aristocracy of Kobaïa. It is a highly refined and complex language, designed to minimize ambiguity through the maximal use of morphological richness. There are 74 cases divided in six families; the family of transrelative cases contains, alone, the cases *oblique, absolutive, dative, ergative, effectuative, inducive, affective, instrumental, activative, derivative, and situative.* There are 7 tones (strictly ascending, mid-ascending, ascending, void, descending, mid-descending, strictly descending); the part of speech which corresponds to nouns and verbs in Earthling languages is called a *formative*: the generation of a formative follows the scheme

$$(((Cv +) VL +) Cg/Cs +) Vr + (Cx/Cv + Vp/VL +) Cr + Vc (+ Ci + Vi) + Ca (+ VxC ) [+ tone] [+ stress]$$

where a central root  $\mathbf{Cr}$  can be declined into a case  $\mathbf{Vc}$  and other auxiliary or optional morphological markers can be added to mark

- Vr: pattern (the static designation assigns *immutable* names, as opposed to the contextual pattern and the temporary pattern) and function (determines the function of the noun being specified: the four functions are *stative*, *dynamic*, *manifestive*, and *descriptive*).

<sup>&</sup>lt;sup>1</sup>Wötâ flöv zï wökoëhl.: my hovercraft (flöv) is full of eels (lit.: the wideboat of-I is eel-y)

 $<sup>^{2}</sup>$ Hurt zanka föhr åföhr: sunrise glows brightly (lit.: first sun shines in a shiny manner).

<sup>&</sup>lt;sup>3</sup> Tâ, nê, zî correspond to I, you, he/she/it, and plural was formed by repeating the singular, tâtâ, nênê, zîzî, although there are attested shortened forms tât, nên, zîz.

- Ca: specifies one among nine possible configurations of the object defined (a single contextual unit embodying the stem concept; an aggregation of incoherent objects of diverse kind; a grouping or set of the basic stem units...), four possible affiliations (consolidative, associative, variative, and coalescent), four possible perspectives (monadic, unbounded, nomic, and abstract), six possible extensions (delimitive, proximal, inceptive, terminative, depletive, and graduative) and two possible essences (normal and representative).
- tone and stress: one of the seven tones and a stress (accent can fall on the ultimate, penultimate, or antepenultimate vowel).

Provide evidence that each language of Kobaïa behaves like a category **PKob**, **SKob**, and **IKob**, with terms of the language as objects and grammatical transformations as morphisms. The usual way to do this (but there might be others, explore them) is to take a subset  $L \subseteq A^*$  of 'grammatically correct' words in an alphabet A (call it, evocatively, a *language* obtained from A) and then consider a relation  $R \subseteq L \times L$  of 'production' of a word w into a word w'. Qua relation, R gives rise to a category taking its (reflexive and) transitive closure  $R^* := \Delta + R + R \circ R + \ldots R \circ \cdots \circ R + \ldots$  For example, if N(Noun), D (aDjective) and V (Verb) are the names of three parts of speech, one can introduce a relation R through the generators

$$N \to ND, \qquad N \to DN$$
  
 $N_1 \to N_1 V N_2, \qquad N_2 \to N_1 V N_2$ 

saying that from a token N (flöv, hovercraft) one obtains a token ND (flöv wökoëhl, eel-y hovercraft), and from nouns  $N_1, N_2$  one obtains sentences  $N_1VN_2$  by attaching  $N_1V$ - or  $-VN_2$  to either the left or right.

Is it possible to build functors of 'poietization'

## $P: \mathbf{IKob} \longrightarrow \mathbf{PKob}$

working as dictionary from imperial to poetic Kobaïan? Is it possible to define a functor that *creates* morphological features, i.e. an 'imperialization' procedure

$$I : \mathbf{PKob} \longrightarrow \mathbf{IKob}$$

to make artists and aristocrat communicate effectively between each other?