Category Theory (ITI9200) – Exercise Sheet 2

Outline. Complete as many of the exercises below as you are able. Each exercise has a number of tasks. Each task has an assigned number of points in square brackets, e.g. [1]. Points may be awarded for answers that demonstrate effort, even if the answer is not entirely correct. There are 25 total points (3 are points for overachievers, the exercise is marked as $[3]^*$). The exercise sheet is expected to take around 2 - 4 hours.

Submission. Email your work to Fosco Loregian at

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or hand in your work to one of the lecturers or teaching assistants at the start of end of a lecture. Deadline:

23:59 on April 18, 2024.

Exercise 1

Let **Set** denote the category of sets and total functions. Define the morphism part, and verify that the following correspondences given at the level of objects define functors $\mathbf{Set} \rightarrow \mathbf{Set}$ (show that the functor laws are satisfied):

- fix a set A; send a set X to $A + (X \times A)$ [1]
- send a set X to $PX \times \{0,1\}$ where PX is the set of subsets of X, and $\{0,1\}$ is a set of Boolean values. [1]

Define the morphism part, and verify that the following correspondences define functors (show that the functor laws are satisfied):

- Let \mathcal{C} be the category where objects are pairs (X, R): X is a set, $R \subseteq X \times X$ a relation on X. Morphisms in $\mathcal{C}((X, R), (Y, S))$ are the functions $f : X \to Y$ such that if $(x, x') \in R$ then $(fx, fx') \in S$. Define a functor $\Gamma : \mathcal{C} \to \mathbf{Graph}$ sending an object (X, R) of \mathcal{C} to the graph having X as set of vertices, and an edge $x \rightsquigarrow x'$ if and only if $(x, x') \in R$. [2]
- Let **Graph** be the category of graphs and graph homomorphisms. If \mathcal{G} is a graph, define the set $Sub(\mathcal{G})$ as the set of *subgraphs* of $\mathcal{G} = (\mathcal{G}_1 \stackrel{s}{\xrightarrow{}} \mathcal{G}_0)$, i.e. the graphs \mathcal{H} such that
 - $-\mathcal{H}_0$ (the vertices of \mathcal{H}) is a subset of \mathcal{G}_0 , and \mathcal{H}_1 (the edges of \mathcal{H}) is a subset of \mathcal{G}_1 ;
 - for every $e \in \mathcal{H}_1$, $s(e), t(e) \in \mathcal{H}_0$.

Sending \mathcal{G} to the set of all its subgraphs $\mathcal{H} \subseteq \mathcal{G}$ is a functor Sub: **Graph**^{op} \rightarrow **Set**. [2]

(Trying to engage with this last question is totally optional, but if you try and get it wrong, there will be no negative repercussion. Do your best!)

Find a graph \mathcal{W} such that $Sub(\mathcal{G})$ is parametrically isomorphic to the set $\mathbf{Graph}(\mathcal{G}, \mathcal{W})$. [3]*

Exercise 2

Let A, B, C be sets. Define functions in opposite directions:

$$A \times (B+C) \xrightarrow{p} A \times B + A \times C \qquad A \times B + A \times C \xrightarrow{q} A \times (B+C)$$

that are inverse to each other: $q \circ p = 1_{A \times (B+C)}$ and $p \circ q = 1_{A \times B+A \times C}$ (verify that these identities hold after defining p, q). [2]

In any category \mathcal{C} with products and sums, it is possible to define a morphism

$$A \times B + A \times C \to A \times (B + C)$$

for each triple of objects $A, B, C \in C_0$. C is called **distributive** if this morphism is an isomorphism for all $A, B, C \in C_0$. Verify that in a distributive category C that admits an initial object \emptyset , the product $A \times \emptyset$ is an initial object (verify that $A \times \emptyset$ has the universal property of an initial object). [2]

Exercise 3

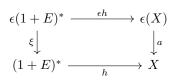
Let **Set** be the category of sets and total functions, and E a fixed set (a set of 'errors').

- Recalling that Maybe(A) = 1 + A denotes the sum of the sets A and 1 = {*}, use distributivity to expand the definition of the correspondence ε : Set → Set sending a set X of 'states' to the set Maybe(X × Maybe(E)).
- Verify in detail that ϵ is a functor.
- An ' ϵ -algebra' is a function

 $\epsilon(X) \xrightarrow{a} X.$

Give an intuition for what a ϵ -algebra might represent, i.e. "It's a function sending an input in ϵX to..." [2]

Prove that the set $(1 + E)^* = \text{List}(\text{Maybe}(E))$ has a structure of an ϵ -algebra $\xi : \epsilon(1 + E)^* \to (1 + E)^*$, and that, for every ϵ -algebra (X, a), there exists a unique function $h : (1 + E)^* \to X$ such that



is a commutative square of functions.

Exercise 4

Let **Dyn** be the category having

- objects the triples (X, x_0, f) where X is a set, $x_0 \in X$ is an element, and $f: X \to X$ is a function;
- a morphism $(X, x_0, f) \to (Y, y_0, g)$ is a function $h: X \to Y$ such that
 - $-h(x_0)=y_0;$

 $-h \circ f = g \circ h$ (which means, the two composed functions coincide input-wise);

• composition and identities are defined as composition of functions in **Set**, as expected.

The category **Dyn** has an initial object (\mathbb{N}, z, s) where N is the set of natural numbers $\{0, 1, 2, ...\}$. What are z and s? Show that this choice of z and s does define an initial object. [2]

Verify that the category **Dyn** is the category of algebras for the endofunctor $Maybe(-) : Set \to Set$ (where Maybe(X) = 1 + X). [1]

Show that we can use the fact that (\mathbb{N}, z, s) is initial to define functions, for each fixed $p \in \mathbb{N}$,

$$\operatorname{plus}_p : N \to N$$

 $\operatorname{times}_p : N \to N$

such that, for all natural numbers $x \in \mathbb{N}$, the following equations hold:

$$plus_p(x) = p + x$$
$$times_p(x) = p \times x$$

[2]

[1]