The universal property of the coKleisli-Kleisli adjunction

Fosco Loregian

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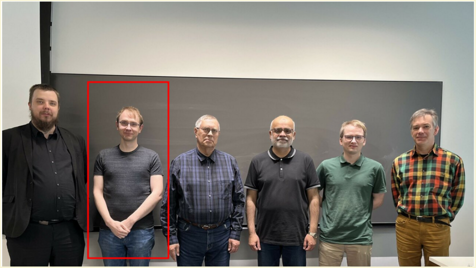
Setting the stage

This work does one thing and tries to do it well.

It’s a (almost completed) work in progress with Nathanael Arkor (TTU) and Ülo Reimaa (UT).



The effect-behaviour adjunction 

Arkor principle: ‘we should discourage the practice of naming theorems or definitions after people’.

Let $\mathcal{C}, \mathcal{D}$ be categories, $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a pair of adjoint functors. There exists an adjunction

$$\hat{G} : \mathbf{coKl}(FG) \rightleftarrows \mathbf{Kl}(GF) : \hat{F}$$

where the left $\hat{G}$ acts on objects like the right G, and

$$\hat{G}(FGX \to Y) = GX \xrightarrow{\eta GX} GFGX \xrightarrow{Gf} GY \xrightarrow{\eta GY} GFGY$$

Similarly, one defines $\hat{F}$ and proves

$$\mathbf{Kl}(FG)(\hat{G}X, Y) \cong \mathbf{coKl}(GF)(X, \hat{F}Y).$$

The effect-behaviour adjunction

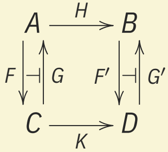
One can apply this construction to

- a Galois connection $f : P \rightleftarrows Q : g$; (the adjunction is an equivalence)
- freely adjoining a basepoint $+1 : \mathbf{Set} \rightleftarrows \mathbf{Set}_* : U$;
- the codomain fibration of a Cartesian category $\times A : \mathbf{Set} \rightleftarrows \mathbf{Set}/A : c$;
- (insert here your favourite adjunction)

Sometimes the result is an interesting adjunction, sometimes it isn't... what's going on? Why the reversal?

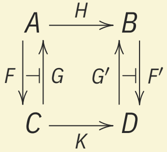
Contramaps of adjoints

- [McL]: a category of adjoints; objects are adjunctions, morphisms are squares compatible with both adjoints.



$$F'H = KF$$
$$G'K = HG$$

- There are also contramaps of adjoints:



$$F'H = KF$$
$$G'K = HG$$

(notation: $\mathcal{C}^{\dagger}(X, Y)$ for contramaps in a 2-category with contravariance)

Classifying contravariance via EB

The EB construction is a functor $\mathbf{Adj} \longrightarrow \mathbf{Adj}$

equipped with a 'canonical' contramap (an adjunction in fact!)

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\textit{free}} & \mathbf{Kl}(GF) \\
F \downarrow \dashv \uparrow G & & \hat{G} \uparrow \dashv \downarrow \hat{F} \\
\mathcal{D} & \xrightarrow[\textit{cofree}]{} & \mathbf{coKl}(FG)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\textit{forget}} & \mathbf{Kl}(GF) \\
F \downarrow \dashv \uparrow G & & \hat{G} \uparrow \dashv \downarrow \hat{F} \\
\mathcal{D} & \xleftarrow[\textit{forget}]{} & \mathbf{coKl}(FG)
\end{array}
$$

such that $\mathbf{Adj}^{\dagger}\big((F \dashv G), (L \dashv R)\big) \cong \mathbf{Adj}\big((\hat{G} \dashv \hat{F}), (L \dashv R)\big)$.

So, the EB construction classifies contravariance.

Idempotency via EB

Theorem

The following conditions are equivalent:

- ▷ $F \dashv G$ *is an* **idempotent** *adjunction;*
- ▷ $\hat{G} \dashv \hat{F}$ *is an* **idempotent** *adjunction;*
- ▷ $\hat{G} \dashv \hat{F}$ *is an* **equivalence** *of categories.*

So, the EB adjunction detects idempotency.

(That's why a Galois connection $f : P \rightleftarrows Q : g$ induces an equivalence: all Galois connections are idempotent...)

So what is this?

What is the EB construction?

Does it have a universal property explaining the previous facts?

Is it (for example) an adjoint to something?

Whatever is going on is certainly 2-dimensional. For quite some time we attempted to explain this construction bicategorically,

Definition 7.3. The 3-equipment of 2-profunctors: objects: 2-categories, vertical 2-category: pseudo-functors; horizontal: pseudo-profunctors.

Definition A.1. Let $\mathscr{V}$ be a multicategory. A *locally $\mathscr{V}$-enriched virtual double category* $\mathbb{X}$ comprises the following data.

but things were, if anything, only getting harder.

In terms of double categories, instead, the nature of the effect behaviour adjunction ‘becomes apparent in terms of a universal construction’. 

Double categories and loose monads

I will gladly skip this slide and save 2 minutes...

A (pseudo)double category $\mathfrak{D}$ is a (pseudo)category internal to **Cat**; it is made of tight arrows (vertical), loose arrows (horizontal) and cells (squares);

$$\begin{array}{ccc} A & \overset{p}{\longrightarrow\!\!\!\!\!\!\!\!+\ \ \ } & B \\ {\scriptstyle f}\downarrow & \alpha & \downarrow{\scriptstyle g} \\ C & \underset{q}{\longrightarrow\!\!\!\!\!\!\!\!+\ \ \ } & D \end{array}$$

- tight arrows compose form the *tight category* $\mathcal{T}\mathfrak{D}$ of $\mathfrak{D}$;
- loose arrows compose ‘up to iso’ so in particular $\mathfrak{D}$ contains the *loose bicategory* $\mathcal{L}\mathfrak{D}$.

Double categories and loose monads

But if you don't know what is a double category:

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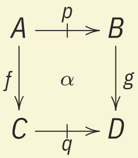

- ▷ tight arrows compose form the *tight category* $\mathcal{T}\mathfrak{D}$ of $\mathfrak{D}$;
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Double categories and loose monads

The double category $\mathfrak{Dist}$ of distributors has cells

$$\begin{array}{ccc} \mathcal{A} & \overset{p}{\nrightarrow} & \mathcal{B} \\ {\scriptstyle F}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle G} \\ \mathcal{C} & \underset{q}{\nrightarrow} & \mathcal{D} \end{array} \qquad \begin{array}{l} p : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Set} \\ \\ q : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set} \end{array}$$

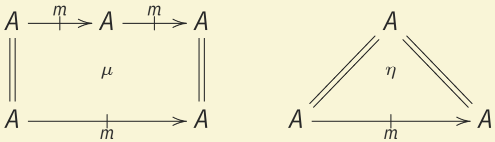
the natural transformations $\alpha : p \Rightarrow q(F, G)$.

To every functor $f : \mathcal{C} \to \mathcal{D}$ one can associate two distributors

- the representable ('companion' of f) $f_* : \mathcal{C} \nrightarrow \mathcal{D}$
  $f_*(d, c) := \mathcal{D}(d, fc)$;
- the corepresentable ('conjoint' of f) $f^* : \mathcal{D} \nrightarrow \mathcal{C}$
  $f^*(c, d) := \mathcal{D}(fc, d)$.

Double categories and loose monads

A loose monad in a double category, over an object A comprises a loose endoarrow $M : A \to A$ together with cells for multiplication and unit ‘satisfying monad axioms’.



- associativity for μ: $\frac{\mu|1_m}{\mu} = \frac{1_m|\mu}{\mu}$;
- left unit: $\frac{\eta|1_m}{\mu} = 1_m$;
- right unit: $\frac{m|\eta}{\mu} = 1_m$.

A loose monad in $\mathfrak{Dist}$ is sometimes called a promonad. If t is a monad on **Cat**, t_* is a loose monad; if s is a comonad on **Cat**, s^* is a loose monad.

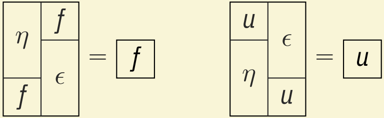
Tight adjunctions

In a double category adjunctions can run in both directions, loose and tight.
A tight adjunction comprises

- tight arrows $f : A \rightleftarrows B : u$
- cells of unit and counit,

$$\begin{array}{ccc} A & \overset{}{=\!\!+\!\!=} & A \\ \Vert & & \downarrow f \\ \Vert & \eta & B \\ \Vert & & \downarrow u \\ A & =\!\!+\!\!= & A \end{array} \qquad \begin{array}{ccc} B & =\!\!+\!\!= & B \\ u \downarrow & & \Vert \\ A & \epsilon & \Vert \\ f \downarrow & & \Vert \\ B & =\!\!+\!\!= & B \end{array}$$

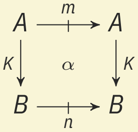
- satisfying adjunction equations:



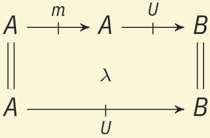
The double category of monads/modules

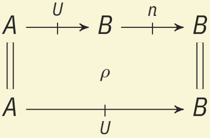
The double category $\mathfrak{Mod}(\mathfrak{Dist})$ of modules has

- objects the loose monads, pairs $(\mathcal{A}, m)$ as before;
- tight arrows the intertwiners $H : A \to B$, equipped with a cell



- loose arrows the bimodules, distributors $U : \mathcal{A} \nrightarrow \mathcal{B}$ equipped with 'actions'





- (cells... the slide is too small)

Reifiers

To every loose monad in $\mathfrak{Dist}$ one can associated the reifier:[1]

Definition (Reifier of a loose monad)

The reifier of a loose monad is category $\mathfrak{R}(m)$ having

- objects the same of $\mathcal{A}$;
- arrows $\xi : a \to b$ the elements $\xi \in m(a, b)$.

Lemma

- *If t is a monad in* **Cat**, *the reifier of t_* is the Kleisli category of t; if s is a comonad in* **Cat**, *the reifier of s^* is the coKleisli category of s.*
- *The reifier assembles into a double functor $\mathfrak{R} : \mathfrak{Mod}(\mathfrak{Dist}) \to \mathfrak{Dist}$.*

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[1] Called collapse by others; we believe it *realizes* heteromorphisms into true arrows, hence the name.

Reifiers

More generally a double category can ‘have reifiers of loose monads’ (it’s a cocompleteness property, which tends to characterize ‘Dist-like’ double categories).

Definition (Having reifiers)

A double category $\mathfrak{D}$ has reifiers if the functor

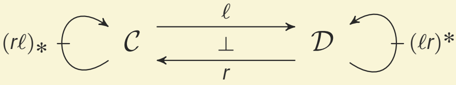
$$\iota : \mathfrak{D} \longrightarrow \mathfrak{Mod}(\mathfrak{D})$$

sending an object to its identity monad has a left adjoint $\mathfrak{R}$.

This is what happens in $\mathfrak{D} = \mathfrak{Dist}$.

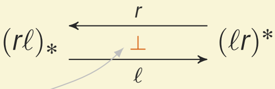
Effect-behaviour, in a double dress

- start with an adjunction $\ell \dashv r$;
- there exists a diagram



where the slashed arrows are considered in $\mathfrak{Dist}$.

- This diagram induces a tight adjunction



(**note the reversal**) in the double category of modules

Effect-behaviour, in a double dress

- apply the reifier to $r \dashv \ell : (\ell r)^* \leftrightarrows (r\ell)_*$: the result is an adjunction (double functors preserve adjunctions!) in the double category of distributors,

$$\mathfrak{R}((r\ell)_*) \underset{\mathfrak{R}\ell}{\overset{\mathfrak{R}r}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathfrak{R}((\ell r)^*)$$

- but now the reifier of the comonad $s^* = (\ell r)^*$ is the coKleisli category $\mathbf{coKl}(s)$ of the comonad, and the reifier of the monad $t_* = (r\ell)_*$ is the Kleisli category $\mathbf{Kl}(t)$!

Q.E.D.: $\hat{G} \equiv \mathfrak{R}r$

In conclusion

A more conceptual perspective:

- a loose monad is a diagram from the 'walking loose monad' double category $\mathfrak{Mnd}$; not a surprise, cf. Bénabou;
- the reifier is the (a kind of) double colimit of the monad-as-diagram $M : \mathfrak{Mnd} \to \mathfrak{D}$;
- the limit of the monad-as-diagram (the 'diagonizer' of M) exists in $\mathfrak{Dist}$ and on (co)representables is the (co)Eilenberg-Moore category of the (co)monad;

Corollary, there is an adjunction between coEilenberg-Moore and Eilenberg-Moore that received more study. In this light, the two constructions are not different from each other and the perfect formal dual of one another.

In conclusion

What next?

Proving that something has a universal property is all fun and games[2] but... what's the big picture here?

That's a good candidate for a question! Just saying... ;-)

[2] Although I believe in the pedagogical value of showing how double categories make things long, but not contrived. One reasons with universal properties, that's all.

In conclusion

A couple of years ago a distinguished professor from Cambridge and our friend Daniele Palombi proposed me to work on the following problem:

Syntax and Models of a non-Associative Composition of Programs and Proofs en fr
Guillaume Munch-Maccagnoni (1, 2)
Show details

- a 'duploid' is something like a category, but composition is not always associative:

$$h \cdot (g \cdot f) = (h \cdot g) \cdot f$$

  if and only if f is a 'thunkable' arrow, or h is a 'linear' arrow (I know... shitty names)

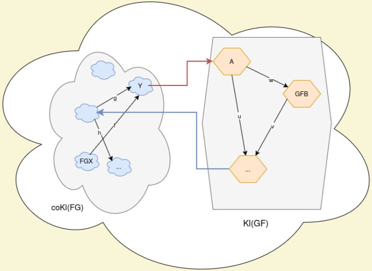
- a duploid is a Very Good Kind™ of virtual double category.

In conclusion

- there is a category of duploids, **reflective** inside the category of adjoints (objects) and adjoint maps (morphisms);
- the effect-behaviour adjunction is essential to build the **reflector**, starting from $F \dashv G$:

In conclusion

- in short, one builds $\hat{G} \dashv \hat{F}$ and then takes a construction like the collage of the profunctor $\mathbf{Kl}(GF)(\hat{G}, 1) \cong \mathbf{coKl}(FG)(1, \hat{F})$, but with heteromorphisms going both ways.



Understanding the universal property of $\hat{G} \dashv \hat{F}$ is an essential step to understand this reflector...

…but that's maybe for next year's ItaCa!

Thank you!

PS: come to Tallinn in July!

9th International Conference on

Applied Category Theory (ACT) 2026

Tallinn, Estonia • 6 July – 10 July

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The 9th International Conference on *Applied Category Theory* will take place in Tallinn, Estonia (venue to be announced) from 6 - 10 July, preceded by the Adjoint School Research Week from June 29 - July 3, 2026. This conference follows previous events at Florida (2025), Oxford (2024), Maryland (2023), Strathclyde (2022), Cambridge (2021), MIT (2020), Oxford (2019) and Leiden (2018).

ACT conference particularly encourages participation from underrepresented groups. The organizers are committed to non-discrimination, equity, and inclusion. The code of conduct for the conference is available here.

Registration

TBA

Important Dates

Abstracts Due	23 March 2026
Full Papers Due	30 March 2026
Author Notification	11 May 2026
Adjoint School	29 June – 3 July 2026
Conference	6 July – 10 July 2026