Two-dimensional Transducers

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Intro

This talk comes from a recent `arXiv` preprint.

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Two-dimensional transducers

Fosco Loregian

We define a bicategory $\mathbf{2TDX}$ whose 1-cells provide a categorification of transducers, computational devices extending finite-state automata with output capabilities. This bicategory is a mathematically interesting object: its objects are categories $\mathcal{A}, \mathcal{B}, \dots$ and its 1-cells $(\mathcal{Q}, t) : \mathcal{A} \to \mathcal{B}$ consist of a category $\mathcal{Q}$ of `states', and a profunctor

$$t : \mathcal{A} \times \mathcal{Q}^{\mathrm{op}} \times \mathcal{Q} \times (\mathcal{B}^*)^{\mathrm{op}} \to \mathbf{Set}$$

where $\mathcal{B}^*$ denotes the free monoidal category over $\mathcal{B}$. Extending t to $\mathcal{A}^*$ in a canonical way, to each `word' $\underline{a}$ in $\mathcal{A}^*$ one attaches an endoprofunctor over the category $\mathcal{Q}$ of states, enriched over presheaves on $\mathcal{B}^*$.
We discuss a number of other characterizations of the hom-category $\mathbf{2TDX}(\mathcal{A}, \mathcal{B})$; we establish a Kleisli-like universal property for $\mathbf{2TDX}(\mathcal{A}, \mathcal{B})$ and explore the connection of $\mathbf{2TDX}$ to other bicategories of computational models, such as Bob Walters' bicategory of `circuits'; it is convenient to regard $\mathbf{2TDX}$ as the loose bicategory of a double category $\mathbb{D}\mathbf{TDX}$: the bicategory (resp., double category) of profunctors is naturally contained in the bicategory (resp., double category) $\mathbf{2TDX}$ (resp., $\mathbb{D}\mathbf{TDX}$); we study the completeness and cocompleteness properties of $\mathbb{D}\mathbf{TDX}$, the existence of companions and conjoints, and we sketch how monads, adjunctions, and other structures/ properties that naturally arise from the definition work in $\mathbb{D}\mathbf{TDX}$.

Comments: Dedicated to Bob Paré, on the occasion of his 80th birthday

arXiv:2509.06769

Intro

It's a systematic (maybe even a bit pedantic) study of transducers, mathematically intended...

- ▷ what's a 'transducer' $(\mathcal{Q}, t)$ categorically?
- ▷ what's the category **TDX** that they form?
- ▷ what mathematics can one do in **TDX**?

💡 Submerge **TDX** inside a bigger 2-category **2TDX**, so **TDX** will be the subcategory spanned by discrete objects.

All (?) will follow.

Intro

People study things called ‘transducers’.

Semibold take

If you’re a category theorist you already know what they are: (instances/generalizations of profunctors)!

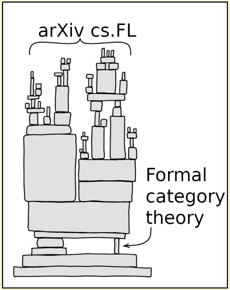
$$\mathcal{A} \overset{\text{relation}}{\nrightarrow} \mathcal{B} \qquad \mathcal{A} \overset{\text{process}}{\longrightarrow} \mathcal{B}$$

How does one make this precise? How close are they to true profunctors? How much more general are they?

Can one think of them representation-theoretically, or in any other compelling way?

Intro

This little work is embedded in a bigger picture:



Formal category theory
as a foundation
for automata theory.

A bit more context is needed now to understand my take on the matter (and what to expect from this talk).

As some of you know, I work in the group of Pawel Sobociński in Tallinn.

> *'You should look into people using category theory to do automata theory! It's probably fun and interesting!'*[a]

[a] 'Plus, you know, the grant who pays you is about that.'

Intro

Now, I don't consider myself a particularly sharp mathematician, but I am very good at malicious compliance.

So I said: ok, there's this line of work I stumbled upon, mentioning 'automata' (a strange word I know near to nothing about) in relation to formal category theory (which I sort of know, and love).

Intro

I can do the latter, and people will think it's the former!

What a delightfully devilish plan. 

So, here I am, a category theorist; I will not pretend I know about applications of this stuff, but I see (more than one) interesting reasons to study these structures as purely mathematical gizmos.

"From automata to bimodules"

DES MACHINES AUX BIMODULES (*)

par René GUITART

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– Introduction.

– § 1 : Calcul des lax-cofins à variables séparées, fibrations et cofibrations.

Calculus of lax-coends in separate variables, fibrations and cofibrations

Completeness for categories of generalized automata

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Appl Categor Struct (2012) 20:251–273
DOI 10.1007/s10485-010-9238-8

Mealy Morphisms of Enriched Categories

Robert Paré

Bicategories of Automata, Automata in Bicategories

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René Guitart

Tenseurs et machines

Cahiers de topologie et géométrie différentielle catégoriques, tome 21, nº 1 (1980), p. 5-62

S. Kasangian
R. Rosebrugh

Decomposition of automata and enriched category theory

Cahiers de topologie et géométrie différentielle catégoriques, tome 27, nº 4 (1986), p. 137-143

René Guitart

Remarques sur les machines et les structures

Cahiers de topologie et géométrie différentielle catégoriques, tome 15, nº 2 (1974), p. 113-144

The semibicategory of Moore automata

Guido Boccali[1], Bojana Femić[2], Andrea Laretto[3], Fosco Loregian[3], and Stefano Luneia[4]

AUTOMATA AND COALGEBRAS IN CATEGORIES OF SPECIES

FOSCO LOREGIAN

Bicategories of processes [1]

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UNA PROPRIETA' DEL COMPORTAMENTO PER GLI AUTOMI COMPLETI (*)

di Renato Betti e Stefano Kasangian (a Milano) (**)

Sommario. - *Si dimostra che, considerando gli automi come rie arricchite, il comportamento è una opfibrazione e aggiunzione con la realizzazione vale anche in questo più generale.*

Summary. - *It is shown that, in the categorical approach by which*

Plan of the talk

Plan of the talk

- ▷ An intuition. What kind of structure is a transducer?
- ▷ A bicategory of transducers.
- ▷ Better yet: a double category.
  - – Properties of the double category. (Limits, colimits, tab, cotab, companions, conjoints)
  - – An interesting point is that tabulators do not all exist.
- ▷ Some things that one can do with this double category.

Transducers and 2-transducers

What is a transducer

Let $\mathcal{A}, \mathcal{B}$ be categories; a 2-transducer is a pair

$$(\mathcal{Q}, t : \mathcal{A} \times \mathcal{Q}^{\text{op}} \times \mathcal{Q} \to \textsf{Set}^{(\mathcal{B}^*)^{\text{op}}}).$$

In particular, if $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ are discrete (=sets), a (1-)transducer is a function

$$t : A \times Q \times Q \longrightarrow 2^{B^*}$$

it's a translator (from words in language A =Arabic, to words in language B =Bengali).

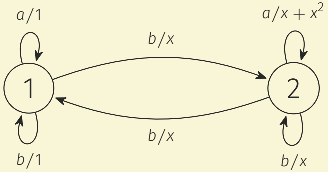
or in other words, a representation

$$t : (A^*, +\!\!+, [\ ]) \longrightarrow (\text{Mat}(Q, 2^{B^*}), \circ, \text{Id}_Q).$$

For example, let $A = \{a, b\}, B = \{x\}$ denote the alphabets, $Q = \{1, 2\}$ the state space. A transducer is specified in terms of two 2-by-2 matrices, say

What is a transducer

To such an arrangement one associates a directed graph representing the *dynamics* of t, where there is a node for each state $q : Q$ and an edge labeled $q \xrightarrow{i/f} q'$, decorated by $(a, f) : A \times 2^{B^*}$ if and only if $t(a)_{qq'} = f$: for the $t(a), t(b)$ above, we then have the diagram



$(\star)$

What is a transducer

Now, it's clear how to interpret the diagram in terms of an elementary notion:

First regard a representation t of A^* as a functor $\Sigma A^* \to \mathbf{Set}$ out of the monoid A^*, regarded as a single-object free category ΣA^*;

Then turn t into an opfibration $P_t : \mathcal{E}[t] \to \Sigma A^*$, via the Grothendieck-Bénabou construction.

> $\mathcal{E}[t]$ *is the free category on the graph* $(\star)$.

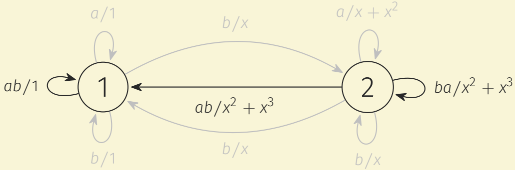
What is a transducer

But this is not the end of the story.

By the UP of free monoids, every transducer t extends on words $w : \{a, b\}^*$; in the example above,to the words 'ab' and 'ba' one associates the product matrices

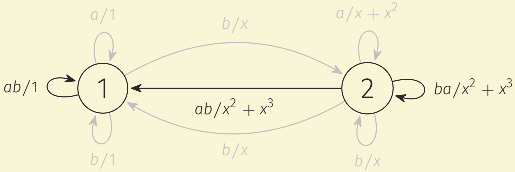
$$t(ab) = t(a)t(b) = \begin{pmatrix} 1 & x \\ x^2+x^3 & x^2+x^3 \end{pmatrix} \qquad t(ba) = t(b)t(a) = \begin{pmatrix} 1 & x^2+x^3 \\ x & x^2+x^3 \end{pmatrix}$$

and thus additional edges appear in the above diagram:



What is a transducer

Keeping the same picture in mind:



it appears that the action category $\mathcal{E}[t]$ is also *graded*:

- the composition of edges $① \xrightarrow{a/1} ① \xrightarrow{b/1} ①$ is the edge $① \xrightarrow{ab/1\cdot 1} ①$ arising as product of labels;
- same for the composition $② \xrightarrow{a/x+x^2} ② \xrightarrow{b/x} ①$, labeled by $ab\big/(x+x^2)\cdot x$.

2-transducers

Categorification (colloquially known as 'the left adjoint to the process of forgetting category theory') now provides the definition we started from: start from

Definition

A 1-transducer is a function

$$t : A \times Q \times Q \times B^* \longrightarrow 2$$

where B^* is the free monoid on B. Currying-extending t, one gets a map $A^* \to Mat(Q, 2^{B^*})$ representing A on the Q-by-Q matrices valued in the free quantale on B.

and categorify the highlighted terms.

2-transducers

Definition

A 2-transducer is a functor

$$t : \mathcal{A} \times \mathcal{Q}^{\mathsf{op}} \times \mathcal{Q} \times (\mathcal{B}^*)^{\mathsf{op}} \longrightarrow \mathbf{Set}$$

where $\mathcal{B}^*$ is the free monoidal category on $\mathcal{B}$.

Currying-extending t, one gets a map $\mathcal{A}^* \to \mathsf{K}\langle\langle\mathcal{B}\rangle\rangle\text{-}\mathbf{Prof}(\mathcal{Q}, \mathcal{Q})$ representing $\mathcal{A}$ on the category of endoprofunctors of $\mathcal{Q}$, enriched in the free (cocomplete) 2-rig $\mathsf{K}\langle\langle\mathcal{B}\rangle\rangle := [(\mathcal{B}^*)^{\mathsf{op}}, \mathbf{Set}]$ on B.

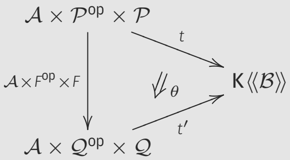
This is the level of generality we will maintain from now on.

2-transducers

Definition

There is a bicategory having

- objects small categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$;
- 1-cells the transducers, pairs $(\mathcal{Q}, t) : \mathcal{A} \to\!\!\!\bullet\!\!\!\to \mathcal{B}$ as above;
- 2-cells $(\mathcal{P}, t) \to (\mathcal{Q}, t')$ the pairs (F, θ) where $F : \mathcal{P} \to \mathcal{Q}$ is a functor, and $\theta : t \Rightarrow t'(F, F)$ fills



In this sense, transducers form a bicategory of profunctors, indexed over the domain, and enriched over the codomain.

2-transducers

Definition

Composition of 1-cells is defined, given 2-transducers $(s, \mathcal{Q}) : \mathcal{A} \multimap \mathcal{B}$ and $(t, \mathcal{P}) : \mathcal{B} \multimap \mathcal{C}$, as the 2-transducer $(\mathcal{P} \times \mathcal{Q}, T \circ (s \times \mathcal{P}^{op} \times \mathcal{P}))$ obtained from the composition

$$\mathcal{A}^* \times \mathcal{Q}^{op} \times \mathcal{Q} \times \mathcal{P}^{op} \times \mathcal{P} \xrightarrow{s \times \mathcal{P}^{op} \times \mathcal{P}} \mathsf{K}\langle\langle \mathcal{B} \rangle\rangle \times \mathcal{P}^{op} \times \mathcal{P} \xrightarrow{T} \mathsf{K}\langle\langle \mathcal{C} \rangle\rangle$$

where $T := \mathsf{Lan}_{y_{\mathcal{B}}} t^*$ is the Laurent-Yoneda ('LY') extension of t (t is first extended to $\mathcal{B}^*$, and then to its presheaves).

Composition of 1-cells is graded: $(\mathcal{P}, \_) \circ (\mathcal{Q}, \_) = (\mathcal{P} \times \mathcal{Q}, \_)$

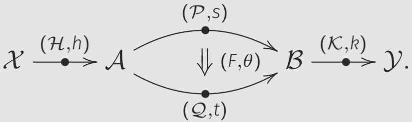
Remark

The identity $\mathcal{A} \multimap \mathcal{A}$ is the pair $(1, \mathsf{hom}_{\mathcal{A}^*})$.

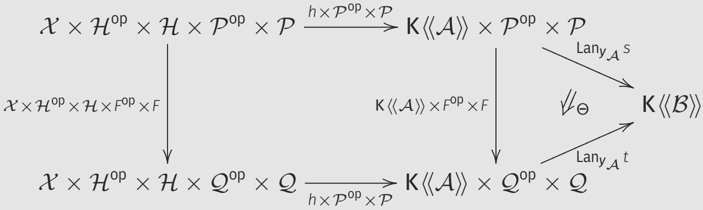
Whiskerings

Remark (On whiskerings)

Given



- ▷ One whiskers 2-cell $(F,\theta) : (\mathcal{P},s) \Rightarrow (\mathcal{Q},t)$ on the left pasting 2-cells



Whiskerings

Remark

- One whiskers on the right, $(\mathcal{K}, k) * (F, \theta)$, simply as

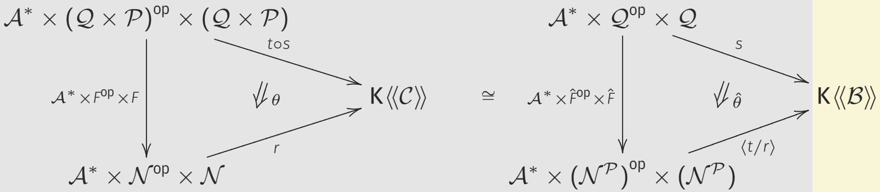
$$
\begin{array}{ccccc}
\mathcal{A} \times \mathcal{P}^{\mathsf{op}} \times \mathcal{P} & & & & \\
& \overset{s}{\searrow} & & & \\
{\scriptstyle \mathcal{A}\times F^{\mathsf{op}}\times F}\big\downarrow & \Downarrow \theta & \mathsf{K}\langle\!\langle \mathcal{B} \rangle\!\rangle & \xrightarrow{K} & \mathcal{V}\text{-}\mathbf{Cat}\big(\mathcal{K}^{\mathsf{op}} \times \mathcal{K}, \mathsf{K}\langle\!\langle \mathcal{Y} \rangle\!\rangle\big) \\
& \underset{t}{\nearrow} & & & \\
\mathcal{A} \times \mathcal{Q}^{\mathsf{op}} \times \mathcal{Q} & & & &
\end{array}
$$

where K is the transpose of $\mathsf{LY}(k) : \mathsf{K}\langle\!\langle \mathcal{B} \rangle\!\rangle \times \mathcal{K}^{\mathsf{op}} \times \mathcal{K} \to \mathsf{K}\langle\!\langle \mathcal{Y} \rangle\!\rangle$.

Adjoints to compositions

Remark (On right extensions/lifts)

Consider two composable 2-transducers $(s, \mathcal{Q}) : \mathcal{A} \twoheadrightarrow \mathcal{B}$ and $(t, \mathcal{P}) : \mathcal{B} \twoheadrightarrow \mathcal{C}$; there is a bijective correspondence between 2-cells (F, θ) of type $t \circ s \Rightarrow r$ and 2-cells $(\hat{F}, \hat{\theta})$ of type $s \Rightarrow \langle t/r \rangle = \mathrm{rift}_t r$,



Similarly, one argues for the existence of right extensions.

1-transducers

As a corollary one deduces the existence of a bicategory of 1-transducers **TDX**:

- objects the sets $A, B, C, D, \ldots$;
- 1-cells $A \longrightarrow\!\!\!\!\!\bullet\!\!\!\!\rightarrow B$ the functions of type

$$t : A^* \times Q \times Q \longrightarrow 2^{B^*}$$

- 2-cells $f : (P, s) \Rightarrow (Q, t)$ the functions $f : P \to Q$ between carriers such that
$$\forall(\underline{a}, p, p') : s(\underline{a}, p, p') \leq t(\underline{a}, fp, fp')$$

Limits and colimits in 2TDX

A double category of transducers

It is more natural to study the (pseudo) double category $\mathbb{D}\mathsf{TDX}$ of which $\mathsf{2TDX}$ is the loose bicategory

Definition

The double category $\mathbb{D}\mathsf{TDX}$ of 2-transducers has

- objects are small categories $\mathcal{A}, \mathcal{B}$, etc.;
- a tight morphism $F : \mathcal{A} \to \mathcal{B}$ is a functor;
- a loose morphism $(\mathcal{Q}, t) : \mathcal{A} \relbar\!\!\bullet\!\!\rightarrow \mathcal{B}$ is a 2-transducer $t : \mathcal{A} \relbar\!\!\bullet\!\!\rightarrow \mathcal{B}$;
- a cell (U, α) with frame $\begin{array}{ccc} \mathcal{A} & \xrightarrow{(\mathcal{Q},s)} & \mathcal{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{A}' & \xrightarrow[(\mathcal{P},t)]{} & \mathcal{B}' \end{array}$ consists of a pair where $U : \mathcal{Q} \to \mathcal{P}$ is a functor and α is a natural transformation with components

$$\alpha : s(a, q, q')(b) \longrightarrow t(Fa, Uq, Uq')(Gb).$$

Co/completeness properties of $\mathbb{D}$TDX

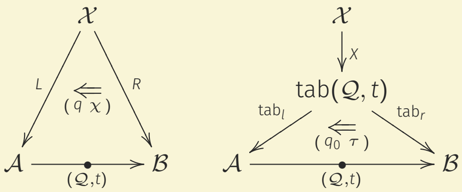
- Coproducts;
- Products;
- Reflexive coequalizers;
- Equalizers;
- cotabulators;
- companions and conjoints

in $\mathbb{D}$TDX exist, similarly to what happens in $\mathbb{P}$rof.

One has to use that the free 2-rig construction is ‘compatible enough’ with the product and coproduct of categories.

The rest of co/limits

Interestingly instead, not all tabulators in $\mathbb{D}$TDX exist: for a 1-cell $(\mathcal{Q}, t) : \mathcal{A} \mathrel{-\!\!\bullet\!\!\to} \mathcal{B}$ to admit a universal factorization



The cell $(\, q \;\; \chi \,)$ must choose an object of $\mathcal{Q}$; but in order for the commutativity above to hold, this object has to be equal to a unique q_0; if $\mathcal{Q}$ does not have a single object, this can't be done.

> *Tabulators where the state category is single object exist (corollary, they do not exist in* TDX *if* $\#Q > 1$*)*

The rest of co/limits

A double category with rich enough co/limits is interesting, but proving co/completeness is not a very conceptually feat.

Instead let me tell you how I would like to look at these gadgets, with a more ‘representation theoretic’ attitude.

A collection of interesting facts

Linear algebra
and transducers

hom-categories between small objects

It is a common theme in defining bicategories of profunctors that hom-categories between ‘small’ objects have a ‘combinatorial’ description.

The bicategory 2TDX makes no exception: $\mathbf{1}$ = terminal category; $\varnothing$ = initial.

- $\mathsf{2TDX}(\varnothing, \varnothing)$, $\mathsf{2TDX}(\mathbf{1}, \varnothing)$, $\mathsf{2TDX}(\mathbf{1}, \mathbf{1})$... are all interesting and interact with each other
- Some of these extend a bit the perspective of ‘profunctors as matrices’
- (A small riddle if you want an exercise: characterise $\mathsf{2TDX}(\{0 \le 1\}, \mathcal{B})$)

On $\mathcal{L} = \mathsf{2TDX}(1, \varnothing)$

A 2-transducer $t : \mathbf{1} \multimap \varnothing$ consists of a pair $(\mathcal{Q}, t)$ where $\mathcal{Q}$ is a category and t a functor of type

$$\mathbf{1} \times \mathcal{Q}^{\mathsf{op}} \times \mathcal{Q} \longrightarrow \mathsf{K}\langle\langle \varnothing \rangle\rangle$$

t extends to a functor $\overset{\mathsf{K}\langle\langle 1 \rangle\rangle}{\mathbf{Set}/\mathbb{N}} \times \mathcal{Q}^{\mathsf{op}} \times \mathcal{Q} \to \mathbf{Set}$ sending $((S_n \mid n : \mathbb{N}), q, q') : \mathbf{Set}/\mathbb{N} \times \mathcal{Q}^{\mathsf{op}} \times \mathcal{Q}$ to

$$T((S_n), q, q') = \sum_{n : \mathbb{N}} S_n \times t^n(q, q').$$

Compare this with the fact that a linear endomorphism $T : V \to V$ yields a $k[X]$-representation acting with a polynomial $g(X) = \sum_n \lambda_n X^i$ on a vector $v : V$ as $\sum_n \lambda_n T^n(v)$.

On $\mathcal{M} = \mathsf{2TDX}(\mathbf{1}, \mathbf{1})$

Let $\mathbf{1}$ be the terminal category, and $\mathcal{M} = \mathsf{2TDX}(\mathbf{1}, \mathbf{1})$ the hom-category of transducers $\mathbf{1} \multimap \mathbf{1}$; clearly, $\mathcal{M}$ is monoidal with respect to composition, and $\mathcal{L}$ above is a $\mathcal{M}$-bimodule.

$\mathsf{2TDX}(\mathbf{1}, \mathbf{1})$ consists of pairs $(\mathcal{Q}, s : \mathcal{Q}^{\mathrm{op}} \times \mathcal{Q} \to \mathbf{Set}/\mathbb{N})$, and acts on an element of $\mathcal{L}$ as composition:

$$\mathcal{Q}^{\mathrm{op}} \times \mathcal{Q} \times \mathcal{P}^{\mathrm{op}} \times \mathcal{P} \xrightarrow{s \times \mathcal{P}^{\mathrm{op}} \times \mathcal{P}} \mathbf{Set}/\mathbb{N} \times \mathcal{P}^{\mathrm{op}} \times \mathcal{P} \xrightarrow{T} \mathbf{Set}$$

sending (q, q', p, p') to $\sum_{n:\mathbb{N}} s(q, q')_n \times t^n(p, p')$.

On $\mathcal{M} = \mathsf{2TDX}(1,1)$

Easy remark: $\mathsf{2TDX}(\varnothing,\varnothing)$ is the category of pairs category $\mathcal{Q}$ / hom functor on $\mathcal{Q}$.

In this light, it is also interesting to work out what the composition map

$$\mathsf{2TDX}(1,\varnothing) \times \mathsf{2TDX}(\varnothing,1) \longrightarrow \mathsf{2TDX}(\varnothing,\varnothing)$$

boils down to: consider two 1-cells

$$\varnothing \xrightarrow{(\mathcal{Q},s)} 1 \xrightarrow{(\mathcal{P},t)} \varnothing$$

the composite 2-transducer $(\mathcal{P},t)\circ(\mathcal{Q},s)$ must be just the hom-functor of a category $\mathcal{C}$, enriched over $\mathsf{Set}/\mathbb{N}$ in the trivial way, i.e. describing the $\mathbb{N}$-graded set constant at $\mathcal{C}(q,q')$.

On $\mathcal{M} = 2\mathrm{TDX}(1,1)$

So, the composition above reduces to

$$\begin{aligned} T((S_n), q, q') &\cong \sum_{n:\mathbb{N}} s(q,q')_n \times t^n(p,p') \\ &\cong \mathcal{Q}(q,q') \times \sum_{n:\mathbb{N}} t^n(p,p') \\ &\cong \mathcal{Q}(q,q') \times \mathcal{P}_{t^*}(p,p') = (\mathcal{Q} \times \mathcal{P}_{t^*})\big((q,p),(q',p')\big) \end{aligned}$$

where $\mathcal{P}_{t^*}$ is the category obtained from the free promonad $\sum_n t^n$ on $\mathcal{P}$, and $\mathcal{Q} \times \mathcal{P}_{t^*}$ the product of categories.

On $\mathcal{M} = \mathrm{2TDX}(1,1)$

This is undeniably a bit mysterious.

Compare with linear algebra: let k be a field; let $\mathbb{N} \pitchfork k \cong \prod_{n:\mathbb{N}} k$ be the *power* of k by $\mathbb{N}$, regarded as a k-algebra.

Every given linear operator $T : V \to V$ of an n-dimensional k-module V is in an evident sense a matrix $[n] \times [n] \to k$, and a sequence of endomorphisms $A_n : W \to W$ of another finite-dimensional k-module W can be regarded as a single matrix $[m] \times [m] \to \mathbb{N} \pitchfork k$; then, one can consider the linear operator on $W \otimes V$, defined as

$$w \otimes v \mapsto \sum_{n \geq 0} A_n w \otimes T^n v$$

provided the sum makes sense; this can be seen as an element of $\mathrm{End}_k(W)[\![T]\!]$ in a suitable sense. The matrix element of $\sum_{n\geq 0} A_n \otimes T^n$ at the entry $((p,q),(p',q'))$ is precisely $\sum_{n\geq 0}(A_n)_{qq'} \otimes (T^n)_{pp'}$.

Monads in $\mathbb{D}$TDX (and 2TDX)

Last, let's peek at loose monads in $\mathbb{D}\mathbf{TDX}$.

'Loose monads in $\mathbb{P}\mathbf{rof}$ are categories.'

Unwinding the definition of monad in $\mathbb{D}\mathbf{TDX}$ one gets:

- A category $\mathcal{A}$ and a functor $\mathcal{A} \times \mathcal{Q}^{\mathrm{op}} \times \mathcal{Q} \times (\mathcal{A}^*)^{\mathrm{op}} \to \mathbf{Set}$;
- Equipping $\mathcal{Q}$ with a monoidal structure $(\boxtimes, j)$;
- So that each $t_{qq'} : \mathcal{A}^* \times (\mathcal{A}^*)^{\mathrm{op}} \to \mathbf{Set}$ is a promonad (i.e., adds heteromorphisms to $\mathcal{A}^*$);
- all done compatibly with a grading

$$t_{uv}(y, z) \times t_{hk}(x, y) \longrightarrow t_{u\boxtimes h, v\boxtimes k}(x, z).$$

Related with multicategories as monads in…

Comparison theorems
with other '2-categories of automata'

KSW ‘processes’

Definition (From ‘Bicategories of processes’)

Let $\mathcal{K}$ be a Cartesian category; the bicategory of Mealy automata is defined as having

- objects, the same of $\mathcal{K}$;
- hom-categories the pullbacks

$$\begin{array}{ccc} \mathbf{Mly}(A,B) & \longrightarrow & (A\times \_/B) \\ \downarrow & & \downarrow U \\ \mathbf{Alg}(A\times \_) & \xrightarrow[U']{} & \mathcal{K}, \end{array}$$

where $\mathbf{Alg}(A\times \_)$ is the category of endofunctor algebras for $A\times - : \mathcal{K}\to\mathcal{K}$, and $(A\times -/B)$ the comma category of arrows $A\times X\to B$ and U, U' are forgetful functors.

$\mathbf{Mly}(A,B) = \{X \leftarrow A\times X \to B\} +$ morphisms between state spaces

KSW ‘processes’

Theorem

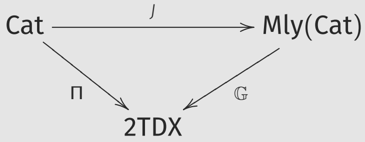
There exists a comparison functor

$$\mathbb{G} : \mathsf{Mly}(\mathsf{Cat}) \longrightarrow \mathsf{2TDX}$$

when $\mathcal{K} = \mathsf{Cat}$*, restricting to a similar comparison* $\mathbb{G} : \mathsf{Mly}(\mathsf{Set}) \to \mathsf{TDX}$ *when* $\mathcal{K} = \mathsf{Set}$.

Theorem

Moreover, there is a pseudocommutative diagram



where $\Pi = \mathbb{G}J$ *is a proarrow equipment (even if* $\mathbb{G}$ *and* J *separately are not proarrow equipments).*

Guitart ‘machines’

Definition (From ‘Remarques sur les machines’)

MAC is the bicategory having

- objects the small categories $\mathcal{A}, \mathcal{B}, \ldots$;
- as hom-categories $\mathsf{Mac}(\mathcal{A}, \mathcal{B})$ the full subcategory of $\mathsf{Span}(\mathsf{Cat})(\mathcal{A}, \mathcal{B})$ spanned by diagrams

$$\mathcal{A} \xleftarrow{q} \mathcal{E} \xrightarrow{g} \mathcal{B}$$

where q is a discrete opfibration.

‘Bicategories of automata’

Guitart 'machines'

Lemma

There is a local reflection $(\_)^\varphi : \mathsf{MAC} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Prof} : j$, *where the bicategory on the right hand side is profunctors, regarded as two-sided discrete fibrations, inside* $\mathsf{Span}(\mathsf{Cat})$.

Theorem

There is a local reflection $\textstyle\int\_ : \mathsf{2TDX} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Prof} : u$, *induced by the span representation of profunctors.*

In other words, for every $\mathcal{A}, \mathcal{B} : \mathsf{Cat}$ there is a reflection at the level of the hom-sets

$$(\textstyle\int\_)_{\mathcal{A}\mathcal{B}} : \mathsf{2TDX}(\mathcal{A}, \mathcal{B}) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathsf{Prof}(\mathcal{A}, \mathcal{B}^*) : (u\_)_{\mathcal{A}\mathcal{B}}.$$

Two directions for future work

Fully monoidal transducers

On ‘fully monoidal’ transducers

There is motivation to study a double category of monoidal transducers, where instead of $\mathcal{A}^*, \mathcal{B}^*$ one considers possibly nonfree monoidal categories; the typical cell here is (u, α) with frame

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{(\mathcal{Q},s)} & \mathcal{N} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{M}' & \xrightarrow[(\mathcal{P},t)]{} & \mathcal{N}' \end{array}$$

consists of a pair where $u : \mathcal{Q} \to \mathcal{P}$ is a functor and α is a natural transformation with components

$$\alpha : s(a, q, q')(b) \longrightarrow t(Fa, uq, uq')(Gb),$$

On ‘fully monoidal’ transducers

The embedding result of **Mly(Cat)** in such a double category $\mathbb{MD}$**TDX** relies on a more refined compatibility between the output map S of a span

$$\mathcal{X} \xleftarrow{D} \mathcal{M} \times \mathcal{X} \xrightarrow{S} \mathcal{N},$$

of functors, namely the property that there is a natural isomorphism in $\mathcal{N}$,

$$S(M, X) \otimes S(M \ltimes X, M') \xrightarrow{\phi_{XMM'}} S(M \otimes M', X)$$

subject to suitable compatibility conditions.

(‘Monads in bicategories of circuits’ sheds a very partial light on the nature of this condition.)

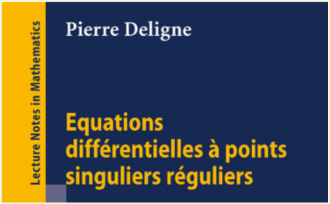
Categorified differential equations?!

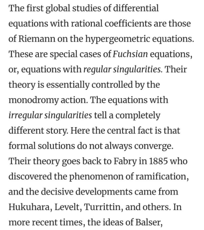
Towards CDE

In the theory of differential equations one is led to study systems of type

$$\dot{y} = Ay$$

where $y = y(z)$ is a n-tuple of differentiable functions, and $A = A(z)$ is a $n \times n$ matrix of functions whose entries are power (or, more generally, Laurent) series.





The first global studies of differential equations with rational coefficients are those of Riemann on the hypergeometric equations. These are special cases of *Fuchsian* equations, or, equations with *regular singularities*. Their theory is essentially controlled by the monodromy action. The equations with *irregular singularities* tell a completely different story. Here the central fact is that formal solutions do not always converge. Their theory goes back to Fabry in 1885 who discovered the phenomenon of ramification, and the decisive developments came from Hukuhara, Levelt, Turrittin, and others. In more recent times, the ideas of Balser,

Towards CDE

Semibold take

Categorification of such a setting passes for a 2-category of 'symmetric transducers'.

Let $\mathcal{Q} = \mathbf{Bij}$ be the category of finite sets and bijections (so the free symmetric monoidal category $\mathbf{1}^\sigma$ on a single generator)

then one can consider the free symmetric 2-rig $\mathcal{V}_\mathcal{B} := \mathsf{K}_\sigma\langle\langle\mathcal{B}\rangle\rangle$ as base of enrichment, and the category of $\mathcal{V}_\mathcal{B}$-enriched combinatorial species having objects

$$Y : \mathbf{Bij}^{\mathrm{op}} \to \mathsf{K}_\sigma\langle\langle\mathcal{B}\rangle\rangle.$$

Towards CDE

Here a ‘differential system’ makes sense:

- given $Y : \mathbf{Bij}^{\mathrm{op}} \to \mathsf{K}_\sigma\langle\!\langle \mathcal{B} \rangle\!\rangle$ (vector) and $A : \mathbf{Bij}^{\mathrm{op}} \times \mathbf{Bij} \to \mathsf{K}_\sigma\langle\!\langle \mathcal{B} \rangle\!\rangle$ (matrix),
- the equation $\partial Y \cong A \otimes Y$ is an isomorphism between the derivative of Y (still of type $\mathbf{Bij}^{\mathrm{op}} \to \mathsf{K}_\sigma\langle\!\langle \mathcal{B} \rangle\!\rangle$),
- and the ‘matrix-vector product’

$$A \otimes Y = \int^{n:\mathbf{Bij}} A(n, \_) \otimes Yn : \mathbf{Bij}^{\mathrm{op}} \to \mathsf{K}_\sigma\langle\!\langle \mathcal{B} \rangle\!\rangle$$

given by profunctor composition.

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Space for doodling