FORMAL CATEGORY THEORY

A COURSE HELD AT MASARYK UNIVERSITY

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Abstract. These are the notes of a reading seminar on formal category theory we are running at Masaryk University. It is not live-TEXed, and yet it is full of misprints, references to nameless cults, blood stains, blasphemy.

1. Ouverture: what is formal category theory

The language of category theory is built upon a certain number of fundamental notions: among these we find the universal characterization of co/limits, the definition of adjunction, (pointwise) Kan extension, and the theory of monads.

It is possible to 'axiomatize' these definitions, pretending that they refer to the 1- and 2-cells of a generic 2-category other than **Cat**. This conceptualization is one of the pillars upon which category theory is done: in some sense, category theory arises when the way in which abstract patterns interact becomes itself an object of study, and when it is generalized to several different contexts. In a few words, the aim of *formal category theory* is to provide a framework in which this process can be outlined mathematically. Quoting the introduction of [**Gra74**], that is one of the pillars on which the subject is founded,

> The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well as to other parts of mathematics. [...]

The basic idea is that the category of small categories, **Cat**, is a 2-category with properties in the same way **Set** is a category with properties. The aim of formal category theory is to outline these properties, and the assumptions needed to ensure that a certain 2-category behaves like **Cat** in some or some other respects.

Unfortunately, being too naïve when performing this process doesn't always give the 'right' answer (by which we mean that it doesn't always build an object with the right universal property), or at least it doesn't give the right answer in the same straightforward way in which some categories of algebraic structures can be defined starting from the category of **Set**.

This is ultimately due to the fact that, when moving to the setting of $\mathcal{V}$-enriched categories (which is the adjacent step of abstraction from **Cat** = **Set**-**Cat**) the theory 'behaves differently' in various ways, and some of these differences prevent $\mathcal{V}$-categories to be as expressive as one would have liked it to be (a paradigmatic example of this minor expressivity is the lack of a *Grothendieck construction* for generic $\mathcal{V}$-presheaves: seeing how the Grothendieck construction, a certain rule to relate 'presheaves on B' and 'fibrations over B', ultimately pertains to formal category theory has been one of the purposes of the early literature on the subject, see [**Str74, SW78, Str80**]).

The major problem is that the 2-category $\mathcal{V}$-**Cat** often doesn't give enough information about the $\mathcal{V}$-valued hom-functors in a 2-category. Formal category

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theory can be thought as a way to encode the same amount of information in various other ways: even though it is always possible to do some constructions by mimicking definitions from **Cat** (adjunctions and adjoint equivalences, extensions by universal 2-cells, etc.), things get a little hairy when we want to provide the theory with an analogue of a very useful and basic result as the Yoneda lemma.

In the 2-category **Cat**, we can use a lax limit construction to “revert” set-valued functors on an object B into arrows “over” B (we basically glue together a bunch of fibers $\coprod_b E_b$ projecting onto B, in the same manner we build the étale space of a presheaf $F : B^{\mathrm{op}} \to \mathbf{Set}$); in the 2-category **Cat** the comma object of $b : 1 \to B$ to $1_B : B \to B$ together with its projection $b/B \to B$ is a good stand in for the covariant functor represented by B (more generally, *discrete left fibrations* over B stand in for general functors $B \to \mathbf{Set}$).

In the 2-category $\mathcal{V}$-**Cat**, we care about $\mathcal{V}$-valued $\mathcal{V}$-functors and we would like to do the same construction there. But for an object b in a $\mathcal{V}$-enriched category B, the comma b/B is more naturally an *internal* category (whose object of objects is $\coprod_{x\in B} B(b, x)$) rather than an *enriched* one (whose objects are morphisms $b \to x$ in the underlying category of B). We have to ensure that the codomain projection $\mathcal{V}$-functor $b/B \to B$ from the enriched version of the comma has a fibration-like properties, and this leaves us with the fundamental problem of formal category theory: *which additional structure on a 2-category $\mathcal{K}$ allows to recognize arrows of $\mathcal{K}$ playing the same rôle of discrete fibrations in* **Cat**, *thus providing with a meaningful notion of Yoneda lemma internal to $\mathcal{K}$?*

It has been in the middle age of southern-emisphere category theory that a certain number of ways to describe such extra structures have been invented: the aim of this first chapters is to give a brief account about three (not unrelated) such attempts. At the moment of writing we count

1.1. **Street and Walters’ “Yoneda structures”.** The definition of Yoneda structure given by Street revolves around the possibility to give a formal counterpart of the Yoneda embedding よ: $A \to \boldsymbol{P}A$ with its universal property. This axiomatization is based on the centrality of the Yoneda lemma ‘internal’ to a 2-category $\mathcal{K}$, that has been defined in a fairly heavy-handed way in Street’s previous [**Str74**]. One of the main achievements of the subsequent [**SW78**] is to obtain an elegant and concise axiomatization stemming almost completely from universal properties.

1.2. **Street’s “fibrational cosmoi”.** A particular case of Yoneda structures, where you ask the pseudo-functor $\boldsymbol{P} : \mathcal{K}^{\mathrm{coop}} \to \mathcal{K}$ characterizing a Yoneda structure to be a right 2-adjoint.

1.3. **Wood’s “proarrow equipments”.** A different and more powerful perspective, where you embed $\mathcal{K}$ into a second 2-category $\mathcal{K}^\star$ with a 2-functor $(\_)_* : \mathcal{K} \to \mathcal{K}^\star$, which is the identity on objects and mimicks the behaviour of the embedding $\mathbf{Cat} \to \mathbf{Prof}$, asking that

- the 2-functor $(\_)_*$ is locally fully faithful;
- for each 1-cell $f : A \to B$ in $\mathcal{K}$, every f_* admits a right adjoint in $\mathcal{K}^\star$.

(See [**Béner, Lor15**] for a thorough account of the theory of profunctors; since the request that $(\_)_*$ is the identity on objects is a bit of an evil one, the paper [**Woo85**] gives the 2-functors satisfying only the other two axioms the name of *pro-equipments*).

2. Yoneda structures on 2-categories

2.1. Lift, extension, contraction, expansion.

Definition 2.1 : Let $B \xrightarrow{f} A \xleftarrow{g} C$ a cospan of 1-cells in $\mathcal{K}$. A *left lifting* of f along g consists of a pair $\langle \mathrm{lift}_g f, \eta\rangle$ (often denoted simply as $\mathrm{lift}_g f$) initial among the commutative triangles like the one below:

$$
\begin{tikzcd}
 & C \arrow[d, "g"] \\
B \arrow[ur, dotted, "\mathrm{lift}_g f"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\Nearrow \eta"] & A
\end{tikzcd}
\qquad
\frac{\mathrm{lift}_g f \to h}{f \to gh}
$$

In other words, composition with $\eta \colon f \Rightarrow g \circ \mathrm{lift}_g f$ determines a bijection $\bar{\gamma} \mapsto (g * \bar{\gamma}) \circ \eta$ between 2-cells $\mathrm{lift}_g f \xrightarrow{\bar{\gamma}} h$ and 2-cells $f \to gh$.

Remark 2.2 : One can define *right liftings* similarly, reversing only the direction of the 2-cell in the diagram above, and consequently the universal property, and *left* and *right extensions* reversing, respectively, only the directions of 1-cells or the direction of both 1- and 2-cells in the diagram above. It is then clear that left extensions in $\mathcal{K}$ are left liftings in $\mathcal{K}^{\mathrm{op}}$, right liftings in $\mathcal{K}$ are left liftings in $\mathcal{K}^{\mathrm{co}}$, and right extensions are left liftings in $\mathcal{K}^{\mathrm{coop}}$.

The situation is conveniently depicted in the following array of universal objects:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, shift left=2ex, "\swarrow\eta"] & B \\
C \arrow[ur, dotted, "\mathrm{Lan}_g f"'] &
\end{tikzcd}
\quad \frac{\mathrm{Lan}_g f \to h}{f \to hg}
\qquad\qquad
\frac{\mathrm{Lift}_g f \to h}{f \to gh}
\begin{tikzcd}
 & C \arrow[d, "g"] \\
B \arrow[ur, dotted, "\mathrm{Lift}_g f"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\Nearrow \eta"] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, shift left=2ex, "\nearrow\varepsilon"] & B \\
C \arrow[ur, dotted, "\mathrm{Ran}_g f"'] &
\end{tikzcd}
\quad \frac{hg \to f}{h \to \mathrm{Ran}_g f}
\qquad\qquad
\frac{h \to \mathrm{Rift}_g f}{gH \to f}
\begin{tikzcd}
 & C \arrow[d, "g"] \\
B \arrow[ur, dotted, "\mathrm{Rift}_g f"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\swarrow \varepsilon"] & A
\end{tikzcd}
$$

Definition 2.3 : There is an obvious notion of *preservation* of a left lifting $\mathrm{lift}_g f$ (write it down abstracting a little bit from the definition of preservation of a co/limit) under the composition with a 1-cell u; we say that a left lifting is *absolute* if it is preserved by *every* u. Of course similar definitions apply to right liftings and left or right extensions.

2.1.1. *Three standard results on lifts and extensions.* Recall that we call

$$f : X \leftrightarrows Y : g$$

a pair of *adjoint* 1-cells if we are given a 2-cell $\epsilon : fg \Rightarrow 1$ and $\eta : 1 \Rightarrow gf$ satisfying the zig-zag identities $g * \varepsilon \circ \eta * g = 1_g$ and $\varepsilon * f \circ f * \eta = 1_f$. We denote this situation in the compact form $f \frac{\eta}{\varepsilon}\!\!\mid g : X \leftrightarrows Y$.

Lemma 2.4 [The most intrinsic characterization of adjointness you could ever think of]**:** The following conditions are equivalent:

(1) f is the left lifting of the identity 1_A along $g : B \to A$ and this lifting is preserved by $g : B \to A$.
(2) f is the absolute left lifting of the identity 1_A along $g : B \to A$;
(3) $f \frac{\eta}{\varepsilon}\!\!\mid g : A \leftrightarrows B$;

Proof. It is obvious that (2) ⇒ (1). We then prove that (1) ⇒ (3) and (3) ⇒ (2).

- The trick is to find $\epsilon : fg \Rightarrow 1$ satisfying the zig-zag identities: since $\langle fg, \eta * g\rangle$ is the left lifting $\mathrm{lift}_g g$, there is a unique such ϵ such that the equation $g * \epsilon \circ \eta * g = 1_g$ holds in the diagram below:

$$\begin{tikzcd} & B \arrow[d, "g"] \\ B \arrow[ur, "fg"] \arrow[ur, bend left=60, "1"] \arrow[ur, phantom, bend left=30, "\Uparrow \epsilon"] \arrow[r, "g"'] \arrow[ur, phantom, bend right=30, "\eta * g"] & A \end{tikzcd}$$

This is one of the zig-zag identities testifying that $f \overset{\eta}{\underset{\epsilon}{\dashv}} g$. The other zig-zag identity can be obtained from the chain of equalities

$$1_a \xrightarrow{\eta} gf \xrightarrow{gf*\eta} gfgf \xrightarrow{g*\varepsilon*f} gf$$

$$1_a \xrightarrow{\eta} gf \xrightarrow{\eta*gf} gfgf \xrightarrow{g*\epsilon*f} gf$$

$$1_a \xrightarrow{\eta} gf =\!=\!=\!=\!=\!=\!= gf$$

but now $\varepsilon * f \circ f * \eta = 1_a$ by uniqueness ($\_ \circ \eta$ induces a bijection $\mathsf{Nat}(gf, gf) \cong \mathsf{Nat}(1, gf)$). This shows that $f \overset{\eta}{\underset{\epsilon}{\dashv}} g$.

- Assuming that $f \overset{\eta}{\underset{\epsilon}{\dashv}} g$, we must show that $\langle f, \eta\rangle$ is an absolute extension. It is an extension, since given a functor $h : X \to A$, and ending the bijections $A(a, gha) \cong B(fa, ha)$ we get that
$$\mathsf{Nat}(1, gh) \cong \mathsf{Nat}(f, h);$$
written explicitly, the bijection sends $\alpha : 1 \Rightarrow gh$ into its *mate* $\tilde{\alpha} = \varepsilon * h \circ F * \alpha : f \Rightarrow fgh \Rightarrow h$, and the uniqueness is given by the bijectivity of $\alpha \mapsto \tilde{\alpha}$. A similar argument shows that this lifting is absolute, as
$$\begin{aligned} \mathsf{Nat}(h, gk) &\cong \int_x A(hx, gkx) \\ &\cong \int_x B(fhx, kx) \\ &\cong \mathsf{Nat}(fh, k); \end{aligned}$$
this shows that $\langle fh, \eta * h\rangle$ is $\langle \mathrm{lift}_g h, \eta * h\rangle$. □

Lemma 2.5 [A PASTING LEMMA FOR LEFT EXTENSIONS]**:** Given the diagram of natural transformations between functors

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[r, phantom, shift right=3, "\eta"] & B \arrow[r, "g"] \arrow[dl, bend left, "k"] & C \arrow[dll, bend left, "n"] \arrow[dl, phantom, "\beta"] \\ D & & \end{tikzcd}$$

assume that the external triangle, plus the left triangle are left extensions, i.e. there are 2-cells $\eta : h \to kf$ such that $k = \mathrm{Lan}_f h$ and $\beta : h \to ngf$ such that $n = \mathrm{Lan}_{gf} h$. Then the right triangle is a left extension, meaning that

- There is a unique $\hat{\beta} : k \to ng$ such that $\beta = \hat{\beta} * f \circ \eta$;
- Such $\hat{\beta}$ makes the pair $(n, \hat{\beta})$ a left extension of k along g.

Proof. Exercise. Use the universal properties you already have to supply the additional one. (Additional question: is it still true for absolute extensions?) □

Definition 2.6 [RELATIVE ADJUNCTION]**:** Let $f : X \to Y$, $j : X \to C$, and $g : Y \to C$ be three functors; we say that f is a *j-relative left adjoint* to g, and we write $f \overset{\eta[j]}{\dashv} g$, if there is a natural isomorphism

$$Y(fx, y) \cong X(jx, gy).$$

Remark 2.7 : It is a matter of checking universal properties to prove that $f \overset{\eta[j]}{\dashv} g$ iff g exhibits $\mathrm{RIFT}_f j$. This gives a formal characterization of relative adjoints generalizing Lemma 2.4.

It is of course possible to define a relative *right* adjoint: given $C \xrightarrow{g} X \xrightarrow{f} Y$ and $j : C \to Y$, we write that $f \underset{[j]\varepsilon}{\dashv} g$ if $Y(fx, jc) \cong X(x, gc)$. Then, $f \underset{[j]\varepsilon}{\dashv} g$ if and only if $f \cong \mathrm{LIFT}_g j$

Remark 2.8 : The definition fo relative adjunction is eminently asymmetric: the most important difference is that the two functors do not determine each other any more, as it is only true that if $f \underset{[j]\varepsilon}{\dashv} g$ then g determines f uniquely, and not viceversa (immediate in view of the formal characterization above).

Apart from this, and keeping in mind this inevitable asymmetry, pretty much all the theory of adjoint functors can be relativized:

- Relative adjunctions have units *or* counits, depending on whether they are left or right;
- relative adjunctions generate *relative monads* in the sense of [];
- if $f \overset{\eta[j]}{\dashv} g$, and $j \overset{\tilde\eta}{\dashv} j_r$ then $f \overset{j_r * \eta'[j_r j]}{\dashv} j_r g$, with η' a pasting of $\eta, \tilde\eta$.

2.2. **Yoneda structures: presenting the axioms.** The idea in this section is to present the bare axioms and then show why these are sensible abstractions of 'trivially true' properties of the 2-category **Cat**. The analogy here is with the motivation for Giraud axioms characterizing a Grothendieck topos, or with the categorical properties of **Set** that characterize (a weak version of) it in ETCS. Then, we discuss the consequences of the axiom we single out showing the most we can in a purely formal way.

We establish the following notation:

- $\mathcal{K}$ is a 2-category, fixed once and for all;
- $\mathsf{Adm}(A, B) \subseteq \mathcal{K}(A, B)$ is a full subcategory of "admissible" 1-cells, which is moreover a *right ideal*, meaning that the composition map restricted to admissible 1-cells gives
$$\mathsf{Adm}(A, B) \times \mathcal{K}(X, A) \to \mathsf{Adm}(X, B).$$
We call admissible an object A such that $1_A \in \mathsf{Adm}(A, A)$; notice that this entails that every 1-cell with admissible codomain is itself admissible.
- we assume that the following structure can be found on $\mathcal{K}$:
  (1) for each admissible object $A \in \mathcal{K}$ we can find an admissible 1-cell $\text{よ}_A : A \to \boldsymbol{P}A$ called a *Yoneda arrow*;
  (2) for each $f : A \to B$ admissible 1-cell with admissible domain, we can find a 2-cell

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "\text{よ}_A"] & \\ B \arrow[r, "B(f,1)"'] \arrow[ur, phantom, "\swarrow\chi^f"] & \boldsymbol{P}A \end{tikzcd}$$

Axiom 1

The pair $\langle B(f, 1), \chi^f \rangle$ exhibits $\mathrm{lan}_f\, \text{よ}_A$.

The validity of this axiom in **Cat** justifies the notation: indeed, in **Cat** the functor $B(f,1)$ amounts precisely to the functor $\lambda b.\lambda a.B(fa,b)$. Of course, a functor is admissible if it has small domain, and $\boldsymbol{P}A$ is the category $[A^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on A.

The proof that $B(f,1) \cong \mathrm{lan}_f \text{よ}_A$, here and elsewhere, will be the result of a nifty coend-juggling: we have that

$$\begin{aligned} \mathrm{lan}_f \text{よ}_A(b) &\cong \int^a B(fa,b) \cdot \text{よ}_A(a) \\ &\cong \int^a B(Fa,b) \times A(\_,a) \\ &\cong B(f\_,b). \end{aligned}$$

Axiom 1 entails that the correspondence $B(f,1)$ is, in a suitable sense, functorial, as a map

$$\mathsf{Adm}(A,B)^{\mathrm{op}} \times \mathcal{K}(X,B) \longrightarrow \mathcal{K}(X,\boldsymbol{P}A)$$

Indeed, given a 2-cell $\alpha : f' \Rightarrow f$ between admissible 1-cells in $\mathcal{K}$, there is a unique $\bar{\alpha} : B(f,1) \Rightarrow B(f',1)$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\text{よ}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow\chi^f"] & \boldsymbol{P}A \\ B \arrow[ur, "B(f,1)"] \arrow[ur, bend right=40, "B(f',1)"'] \arrow[ur, phantom, bend right=20, "\Searrow c"] & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "\text{よ}_A"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\Leftarrow_{\alpha}"] \arrow[dr, phantom, "\swarrow\chi^f"] & \boldsymbol{P}A \\ B \arrow[ur, bend right, "B(f',1)"'] & \end{tikzcd}$$

commutes for a single 2-cell induced by the universal property of $\mathrm{lan}_f \text{よ}_A$, and such 2-cell can quite rightly be called $B(\alpha,1)$ (notice that the pasting $B(f',1) * \alpha \circ \chi^{f'}$ of the right square exists only if $\alpha : f' \Rightarrow f$, so $\lambda\alpha.B(\alpha,1)$ must be contravariant whatever its definition).

Instead, given a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$, we define

$$\begin{tikzcd} X \arrow[r, bend left, "B(f,b)"] \arrow[r, bend right, "B(f',b')"'] \arrow[r, phantom, "\Downarrow B(\alpha,\beta)"] & \boldsymbol{P}A \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \arrow[r, bend left, "B(f,1)"] \arrow[r, bend right, "B(f',1)"'] \arrow[r, phantom, "\Downarrow B(\alpha,1)"] & \boldsymbol{P}A \end{tikzcd}$$

Axiom 2

The pair $\langle f, \chi^f \rangle$ exhibits $\mathrm{lift}_{B(f,1)} \text{よ}_A$.

The validity of this axiom in **Cat** is again a game of coend calculus: if we call $N_f = \mathrm{lan}_f \text{よ}_A = B(f,1)$ for short, we have $\mathrm{lift}_{N_f} \dashv N_{f,*}$, where $N_{f,*} \colon g \mapsto N_f \circ g$ is the 'direct image' functor; then we have

$$\begin{aligned} \mathsf{Nat}\big(\text{よ}_A, N_f \circ g\big) &\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(\text{よ}_A a', N_f \circ g(a')\big) \\ &\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(\text{よ}_A a', B(f\_, ga')\big) \\ &\cong \int_{a'} B(fa', ga') \\ &\cong \mathsf{Nat}(f,g) \end{aligned}$$

Axiom 3

Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells

$$\begin{tikzcd}
A \arrow[rr, "\text{よ}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \chi^{\text{よ}_B f}"] & & \boldsymbol{P}A \\
B \arrow[r, "\text{よ}_B"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow \chi^{g}"] & \boldsymbol{P}B \arrow[ur, "\boldsymbol{P}f"'] & \\
C \arrow[ur, "C(g,1)"'] & &
\end{tikzcd}$$

exhibits $\mathrm{lan}_{gf}\,\text{よ}_A = C(gf,1)$, and the pair $\langle 1_{\boldsymbol{P}A}, 1_{\text{よ}_A}\rangle$ exhibits $\mathrm{lan}_{\text{よ}_A}\,\text{よ}_A$.

The hidden meaning of this axiom is that $\boldsymbol{P}$ is a pseudofunctor $\mathcal{K}^{\mathrm{coop}} \to \mathcal{K}$.

Let's make this evident: given a pair of composable the universal property of χ^{gf} entails that there is a unique 2-cell θ^{gf} filling the diagram

$$\begin{tikzcd}
A \arrow[r, "\text{よ}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow \chi^{gf}"] & \boldsymbol{P}A \\
B \arrow[d, "g"'] & \boldsymbol{P}B \arrow[u, "\boldsymbol{P}f"'] \arrow[ul, phantom, "\theta^{gf} \searrow"] \\
C \arrow[r, "\text{よ}_C"'] \arrow[uur] \arrow[ur] \arrow[r, phantom, shift left=3, "\searrow"] & \boldsymbol{P}C \arrow[u, "\boldsymbol{P}g"']
\end{tikzcd}$$

Axiom 3 is equivalent to the request that this arrow is invertible (exercise: draw the right diagram), and this yields that the above diagram has the same universal property of

$$\begin{tikzcd}
A \arrow[r, "\text{よ}_A"] \arrow[d, "gf"'] \arrow[dr, phantom, "\Downarrow"] & \boldsymbol{P}A \\
C \arrow[r, "\text{よ}_C"'] & \boldsymbol{P}C \arrow[u, "\boldsymbol{P}(gf)"']
\end{tikzcd}$$

which in turn entails that there is a unique, and invertible, 2-cell $\boldsymbol{P}(gf) \Rightarrow \boldsymbol{P}f \circ \boldsymbol{P}g$. This is of course the first part of the structure of pseudofunctor on $\boldsymbol{P}$; the remaining structure is given by the request that $\langle 1_{\boldsymbol{P}A}, 1_{\text{よ}_A}\rangle$ exhibits $\mathrm{lan}_{\text{よ}_A}\,\text{よ}_A$.

Remark 2.9 : As this might appear quite enigmatic, let's recall that we call *dense* a 1-cell k with the property that $\mathrm{lan}_k k \cong 1$; this allows us to rephrase the second part of axiom 3 saying that 'the Yoneda embedding is dense'. This is in fact a characterizing property, as the Yoneda lemma is essentially a statement about the inclusion $A \to \boldsymbol{P}A$ being able to "generate all $\boldsymbol{P}A$ under colimits". As the universal property of $\boldsymbol{P}(1_A)$, defined above, entails that there is a unique 2-cell $\iota_A : \boldsymbol{P}(1_A) \Rightarrow 1_{\boldsymbol{P}A}$, axiom 3 is equivalent to the request that ι_A is invertible (and natural in A). This renders $\boldsymbol{P}$ a pseudo-functor, as claimed above.

All these remarks are of course trivial in **Cat**, since the functoriality of the correspondence $A \mapsto \widehat{A}$ can be proved directly. Nevertheless, axiom 3 is still telling us something about a 'reduction rule' for composition of Kan extensions: indeed, it is possible to prove that (in the same notation of axiom 3)

$$\theta_{gf} : \mathrm{lan}_{gf}\,\text{よ}_A \cong \mathrm{lan}_{\text{よ}_B f}\,\text{よ}_A \circ \mathrm{lan}_g\,\text{よ}_B$$

There is an additional axiom:

Axiom 4

Let $\begin{tikzcd} B \arrow[r, bend left, "B(f,1)"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\sigma"] & \boldsymbol{P}A \end{tikzcd}$ be a 2-cell; if it has the property that the pasting

$$\begin{tikzcd} A \arrow[r, "よ_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow", near start] & \boldsymbol{P}A \\ B \arrow[ur] \arrow[ur, bend right, "g"'] \arrow[ur, phantom, "\Searrow\sigma", near end] & \end{tikzcd}$$

exhibits $\mathrm{lift}_g よ_A$, then σ is invertible.

It must be noted that this axiom is mainly useful to make some statements and proofs look better: there are reasons not to include it in the definition of a bare Yoneda structure; it can be proved that $1, 2, 4 \Rightarrow 3$ so that we can call *nice Yoneda structures* those that satisfy $1, 2$, and 4.

We now concentrate on a few examples that make evident how category theory can be re-enacted in a 2-category with a Yoneda structure. In a nice Yoneda structure, we have a nicer characterization of adjoints and a more intuitive analogue of fully faithful 1-cells:

- If axiom 4 holds, then we recover the characterization of relative adjoints below (see 2.10; in general, only one implication holds) in terms of left liftings: given 1-cells $f : A \to B, g : B \to C, j : A \to C$ we have $f \cong \mathrm{lift}_g j$ if and only if $f \overset{\eta[j]}{\dashv} g$, i.e. if and only if $B(f, 1) \cong C(j, g)$.
- if axiom 4 holds, then a 1-cell f is fully faithful if and only if the *functor* $\mathcal{K}(X, f)$ is fully faithful for each X, naturally in X.

2.3. **Yoneda structures: theorems.** A great deal of category theory can be developed in a category $\mathcal{K}$ endowed with a Yoneda structure. We collect here a few results coming from [**SW78**].

Theorem 2.10 [ON RELATIVE ADJOINTS]**:** Suppose $j : A \to C, f : A \to B, g : B \to C$ are 1-cells in $\mathcal{K}$ forming a relative adjunction $f \overset{\eta[j]}{\dashv} g$, and A, f, j are admissible. Then the equality of 2-cells

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "よ_A"] \arrow[d, phantom, "\Downarrow\chi^f"] & \\ B \arrow[rr, bend left, "B(f,1)"] \arrow[rr, bend right, "C(j,1)\circ g"'] \arrow[rr, phantom, "\Downarrow\pi"] & & \boldsymbol{P}A \end{tikzcd} = \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "j"] \arrow[dr, "よ_A"] & \\ B \arrow[r, "g"'] \arrow[ur, phantom, "\Swarrow\eta", near start] & C \arrow[r, "C(j,1)"'] \arrow[ur, phantom, "\Swarrow\chi^j", near start] & \boldsymbol{P}A \end{tikzcd}$$

holds since the left 2-cell defines a $よ_A \Rightarrow C(j, g) \circ f$, that (by the universal property of χ^f) must be of the form $(\pi * f) \circ \chi^f$ for a unique $\pi : B(f, 1) \Rightarrow C(j, g)$. This determines a bijection

$$\frac{\pi : B(f, 1) \Rightarrow C(j, t)}{\eta : j \Rightarrow gf}$$

If π is invertible then the corresponding η exhibits $\mathrm{LIFT}_g j$.

Theorem 2.11 [ON ADJOINTS]**:** Take $f : A \leftrightarrows B : g$ such that A, f are admissible. Given a 2-cell $\eta : 1 \to gf$, the universal property of $よ_A * \eta$ induces a bijection between these η's and 2-cells $\pi : B(f,1) \Rightarrow A(1,u)$:

$$
\begin{tikzcd}
A \arrow[r, "\text{よ}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow \text{よ}*\eta"] & \boldsymbol{P}A \\
B \arrow[ur, bend right, "A(1,u)"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\text{よ}_A"] \arrow[d, "f"'] & \boldsymbol{P}A \\
B \arrow[ur, "B(f,1)"] \arrow[ur, bend right, "A(1,u)"'] \arrow[ur, phantom, shift right=3, "\overset{\pi}{\Searrow}"] &
\end{tikzcd}
$$

Then η is a unit of an adjunction $f \overset{\eta}{\dashv} g$ if and only if π is invertible. Moreover, if $f \overset{\eta}{\dashv} g$ then for any $X \in \mathcal{K}$, and $a : X \to A, b : X \to B$ 1-cells, wit $X, a, f \circ a$ admissible we have the "pointset" characterization of adjoints $A(a, gb) \cong B(fa, b)$.

Definition 2.12 [WEIGHTED COLIMIT IN FORMAL CATEGORY THEORY]**:** Given admissible $A, f : A \to B$ and $M, j : M \to \boldsymbol{P}A$ we define a j-indexed colimit (or j-weighted colimit) for f, and write $j \otimes f : M \to B$ for a j-relative left adjoint of $B(f,1)$. This means that we can write a "tensor-hom"-like adjunction

$$B(j \otimes f, 1) \cong \boldsymbol{P}A(j, B(f,1))$$

Remark 2.13 : Notice that Thm. 2.10 above characterizes $j \otimes f$ as an absolute left lifting of j along $B(f,1)$; the converse is in general not true (it requires axiom 4), and then this left extension deserves the name of *weak* j-indexed colimit.

In view of the above remark, it is obvious what does it mean for a 1-cell h to *preserve* a j-indexed colimit $j \otimes f$. We have the following

Theorem 2.14 : A left adjoint 1-cell $l : A \to B$ preserves all (weak) j-indexed colimits that exist in $\mathcal{K}$ and can be composed with l.

Proof. The proof is notationally tautological (and shows the power of endowing a 2-category with a Yoneda structure) in view of the definition for the left extension $X(u,1)$ and the composition $X(1,u) = \text{よ}_X u$: assume $l \overset{\eta}{\dashv} r$ is an adjunction with $l : B \to X$, and that the diagram

$$
\begin{tikzcd}
X \arrow[r, "j \otimes f"] \arrow[d, "j"'] \arrow[dr, phantom, "\overset{\eta}{\Nearrow}"] & B \arrow[dl, bend left, "B(f,1)"] \\
\boldsymbol{P}A &
\end{tikzcd}
$$

exhibiting $j \otimes f$ is given; then

$$
\begin{aligned}
\boldsymbol{P}A(j, X(lf,1)) &\cong \boldsymbol{P}A(j, B(f,r)) \\
&= \boldsymbol{P}A(j, B(f,1)) \circ r \\
&\cong B(j \otimes f, 1) \circ r \\
&= B(j \otimes f, r) \\
&\cong X(l(j \otimes f), 1)
\end{aligned}
$$

thus exhibiting $j \otimes lf$. ∎

Theorem 2.15 : Suppose $M, A, j : M \to \boldsymbol{P}A, f : A \to B$ are admissible. If $j \otimes f$ exists and if additional admissible $N, i : N \to \boldsymbol{P}M$ are such that $i \otimes j$ exists, then there is an associativity isomorphism

$$i \otimes (j \otimes f) \cong (i \otimes j) \otimes f.$$

The following theorem is what can be called "ninja Yoneda lemma": it amounts to the statement that tensoring with a representable acts like an evaluation.

Theorem 2.16 [THE NINJA YONEDA LEMMA]**:** For admissible $A, f : A \to B$ and $X, a : X \to A$ there is an isomorphism

$$A(1, a) \otimes f \cong f \circ a$$

2.4. **Yoneda structures: examples.** Here we collect a few examples of Yoneda structures for different choices of $\mathcal{K}$:

(1) the 2-category **Cat** has a Yoneda structure where $\boldsymbol{P}A = [A^{\mathrm{op}}, \mathbf{Set}]$.
(2) the 2-category $\mathcal{V}$-**Cat** of categories enriched over a base $\mathcal{V}$ has a Yoneda structure where $\boldsymbol{P}A = [A^{\mathrm{op}}, \mathcal{V}]$ (you basically pretend your proof lives in **Cat** = **Set**-**Cat**).
(3) the 2-category **Cat**$(\mathcal{K})$ of internal categories in a finitely complete $\mathcal{K}$ (like for example a topos) has a Yoneda structure whose existence we sketch in Exercise 2.24.
(4) If $\mathcal{K}$ has a Yoneda structure and $\mathcal{C}$ is a small 2-category, $\mathrm{Psd}[\mathcal{C}^{\mathrm{op}}, \mathcal{K}]$ has an *objectwise* Yoneda structure.

2.5. **Selected exercises.** This subsection collects a few exercises; the ones labelled with a ☕ symbol are to be done with patience, a comfortable spot in the library and a cup of good coffee (warning: American coffee might be an insufficient adjuvant); the ones labelled with a danger symbol ⚠ are meant to be boring technicalities no one ever checks, or extremely difficult exercises.

Exercise 2.17 : Dualize the statement of Lemma 2.4 and 2.5.

Exercise 2.18 : Let

$$\begin{tikzcd}
(f/g) \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\swarrow\theta"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

be a comma object. Is it true that $\langle p, \theta \rangle$ exhibits $\mathrm{rift}_g fq$?

Exercise 2.19 : Prove that in **Cat** there is an isomorphism

$$\mathrm{lan}_{よ_B \circ f}\, よ_A \cong \mathrm{lan}_{よ_B} B(f, 1)$$

for each $f : A \to B$ a functor between small categories, and the Yoneda embeddings $よ_A : A \to \boldsymbol{P}A$, $よ_B : B \to \boldsymbol{P}B$.

Exercise 2.20 : Prove axiom 3 in **Cat**; write explicitly the isomorphism θ_{fg} of axiom 3.

Exercise 2.21 [☕]**:** Prove that there is a Yoneda structure on the 2-category of posets (objects: posets seen as categories; 1-cells: monotone functions; 2-cells: the partial order relation on the set $\mathbf{Pos}(X, Y)$ of monotone functions).

- Describe explicitly $よ_A : A \to \boldsymbol{P}A$;
- Prove directly axioms 1 and 2;
- Prove axiom 3 pretending to ignore that $\boldsymbol{P}$ is blatanly a functor, i.e. prove directly that θ_{fg} and ι_A are defined and invertible (=identities). Does axiom 4 hold?

Do this without using the description of **Pos**

Exercise 2.22 [⚠]**:** Prove that the isomorphism α that render $\boldsymbol{P}$ a pseudo-functor satisfy the commutativity

$$\begin{tikzcd}
P(hgf) \arrow[r] \arrow[d] & P(gf)Ph \arrow[d] \\
PfP(hg) \arrow[r] & Pf(PgPh) = (PfPg)Ph
\end{tikzcd}$$

(note that this is both boring and difficult: you cna only use the universal property).

Exercise 2.23 : Prove 2.15 and 2.16 in a similar -formal- way of 2.14.

Exercise 2.24 [⚠]**:** Let $\mathcal{E}$ be a finitely complete category, and $\mathcal{K} = \mathbf{Cat}(\mathcal{E})$ the 2-category of categories internal to $\mathcal{E}$. Recall the definition of an internal profunctor [**Bor94**, 8.2.1, 8.4.3]; prove that there is an equivalence

$$\mathbf{Prof}_{\mathcal{E}}(A, B) \cong \mathbf{Prof}_{\mathcal{E}}(1, A^{\mathrm{op}} \times B)$$

Prove that this correspondence is natural in A, B (which covariance type is it?). We define

- an *internal full subcategory* of $\mathcal{E}$ an object $\mathbf{S}$ of $\mathcal{K}$ with an internal profunctor $s : 1 \nrightarrow \mathbf{S}$ inducing a fully faithful functor

$$\mathcal{K}(X, \mathbf{S}) \to \mathbf{Prof}_{\mathcal{E}}(1, B)$$

  via precomposition.
- a 1-cell $f : A \to B$ in $\mathcal{K}$ *admissible* when the profunctor corresponding to (f/B) lies in the essential image of the functor $\mathcal{K}(A^{\mathrm{op}} \times B, \mathbf{S}) \to \mathbf{Prof}_{\mathcal{E}}(1, A^{\mathrm{op}} \times B)$. call f^* this (unique) 1-cell $A^{\mathrm{op}} \times B \to \mathbf{S}$.

Prove that $\mathcal{K}$ has a Yoneda structure when $B(f, 1) := \widehat{f^*} : B \to [A^{\mathrm{op}}, \mathbf{S}]$ is the mate of f^*, and $\boldsymbol{P}A := [A^{\mathrm{op}}, \mathbf{S}]$.

What happens when $\mathcal{E}$ is an elementary topos and $\mathbf{S} = \Omega_{\mathcal{E}}$? What happens when $\mathcal{E}$ is a Grothendieck topos and $\mathbf{S} = \mathbf{N}$ is the natural number object?.

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