Some remarks on the fibration of algebras

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December 21, 2022

- This is a work in progress with G. Coraglia and D. Castelnovo;

- again, this is a work *in progress* and not at all polished.
- Mostly, this talk is a request for help: we don't know how to finish a paper.

Motivation

The simple slice

We want to generalise the following example:

Proposition

Let $\mathcal{C}$ be a cartesian category; we can build the *simple fibration* [Jac99] $\begin{bmatrix} \mathbf{s}(\mathcal{C}) \\ \downarrow \\ \mathcal{C} \end{bmatrix}$ over $\mathcal{C}$, where each fiber $\mathbf{s}(\mathcal{C})_I$ over an object I has

- the same objects of $\mathcal{C}$;
- morphisms $X \times I \to Y$.

Composition of intra-fiber arrows is

$$X \times I \xrightarrow{X \times \Delta} X \times I \times I \xrightarrow{f \times I} Y \times I \xrightarrow{g} Z$$

The category $\mathbf{s}(\mathcal{C})_I$ is called the simple slice $\mathcal{C} /\!\!/ I$.

Motivating examples

A more conceptual on $\mathbf{s}(\mathcal{C})$:

- Consider the comonad $S_I = - \times I$ on $\mathcal{C}$;
- the simple slice $\mathcal{C}/\!\!/I$ is the coKleisli category of S_I;
- composition intra-fiber is coKleisli composition.

So, the simple fibration has some sort of universal property.

Similarly, one can collect the coEilenberg-Moore categories of S_I and obtain a fibration: each $\mathbf{coEM}(S_I)$ is just the slice category over I.

So, the fibration of typical fiber $\mathbf{coEM}(S_I)$ has an even more straightforward universal property.

A few questions

When we started working on this project we had three questions:

- Do we have a theory available of fibrations obtained collecting ‘categories of algebras of parametric endofunctors’?
- if not, can we write it, find more examples, outline what properties are shared by all such fibrations?
- the simple fibration is useful in type theory; how to find a type-theoretic interpretation for (at least some) fibrations of algebras?

The fibration of algebras

The problem

Study and classify fibrations arising from a functor

$$F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$$

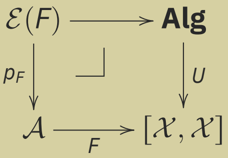
–or its mate $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$ of which we consider objectwise categories of algebras $\mathbf{Alg}(F_I)$, or more precisely:

- consider the prestack $A \mapsto \mathbf{Alg}(F_A)$ as a contravariant functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$;
- this induces a split fibration, under the Grothendieck correspondence, $\begin{bmatrix} \mathcal{E}(F) \\ \downarrow \\ \mathcal{A} \end{bmatrix}$ over the category of parameters.

A starting point

Problem

Study all fibrations p_F arising as pullbacks from a universal fibration of algebras:



- U is the fibration arising from $[\mathcal{X}, \mathcal{X}]^{\mathrm{op}} \to \mathbf{Cat}$;
- all properties of U that are pullback-stable are inherited by p_F, no matter the shape of F.

Define the following fibrations:

- $U : \begin{bmatrix} \mathbf{Alg}_{\mathcal{X}} \\ \downarrow \\ [\mathcal{X},\mathcal{X}] \end{bmatrix}$ with fiber over F the category of endofunctor algebras for F;
- $U_p : \begin{bmatrix} \mathbf{Alg}_{\mathcal{X},p} \\ \downarrow \\ [\mathcal{X},\mathcal{X}]_p \end{bmatrix}$, with fiber over F the category of pointed endofunctor algebras for F;
- $U_m : \begin{bmatrix} \mathbf{Alg}_{\mathcal{X},m} \\ \downarrow \\ [\mathcal{X},\mathcal{X}]_m \end{bmatrix}$ with fiber over a *monad* its category of Eilenberg-Moore algebras.

Morphisms change! $[\mathcal{X},\mathcal{X}]_p$ has natural transformations $\alpha : T \Rightarrow S$ compatible with units; $[\mathcal{X},\mathcal{X}]_m$ has monad morphisms.

Fibrations of algebras

Consider $\left[\begin{smallmatrix} \mathcal{E} \\ p\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ appearing in a pullback like

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathbf{Alg}_{\mathcal{X},\square} \\ {\scriptstyle p}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U_\square} \\ \mathcal{A} & \longrightarrow & [\mathcal{X},\mathcal{X}]_\square \end{array}$$

We say that p is

- an *(endofunctor) algebra fibration* if it fits such a pullback where $U_\square = U$; dualize for coalgebras
- a *pointed algebra fibration* if it fits such a pullback where $U_\square = U_p$; dualize for copointed coalgebras
- an *Eilenberg-Moore fibration* if it fits such a pullback where $U_\square = U_m$ dualize for coEilenberg-Moore.

the Kleisli's version

The case of Kleisli and coKleisli must be treated with a little bit of more care...

...one would love to say that

$$K_T : \mathbf{Kl}_{\mathcal{X}}(T) \to \mathbf{EM}_{\mathcal{X}}(T)$$

assemble into a n.t. $\nu : [\mathcal{X}, \mathcal{X}]_m \underset{\mathbf{EM}_{\mathcal{X}}}{\overset{\mathbf{Kl}_{\mathcal{X}}}{\Downarrow}} \mathbf{Cat}$ of sorts, maybe even fibered over $[\mathcal{X}, \mathcal{X}]_m$...

...but a moment of reflection shows that the statement doesn't even typechecks.

Examples

Examples zero

Clearly, our language is engineered to recover the starting motivating examples:

- the simple fibration arises as the coKleisli fibration of the functor $\mathcal{C} \to [\mathcal{C}, \mathcal{C}] : A \mapsto - \times A$;
- the domain fibration arises as the coEilenberg-Moore fibration of the same functor;
- When A runs over *internal monoids* in $\mathcal{C}$, $- \times A$ is also a *monad* (with algebras A-modules); the associated fibration is the fibration of modules over $\mathbf{Mon}(\mathcal{C})$, an old friend of algebraic geometers/homotopists.

Topologies

- (toy) consider a set X and its poset $T(X)$ of topologies; $\tau \mapsto \mathbf{Sh}(X, \tau)$ defines a fibration over $T(X)$ collecting all sheaves with respect to various topologies;
- (toy for grownups) same, but with Grothendieck sites and site homomorphisms on a given small category $\mathcal{C}$;
- consider the Kelly-Lawvere lattice P_{KL} of a topos $\mathcal{E}$, whose elements are called levels of $\mathcal{E}$ in [KL89]; this defined a fibration over P_{KL} whose typical fiber is an essential localization of $\mathcal{E}$.

Polynomials i

There are two flavours of polynomial functors that our formalism captures:

- ‘classic’ polynomials à la Moerdijk-Palmgren [MP00]: given a locally cartesian closed pretopos $\mathcal{E}$ and an object $f : X \to A$ of the slice $\mathcal{E}/A$ we can define a polynomial endofunctor on $\mathcal{E}$

$$P_f : \mathcal{E} \xrightarrow{\pi} \mathcal{E}/A \xrightarrow{\langle f, - \rangle} \mathcal{E}/A \xrightarrow{s} \mathcal{E}$$

where f plays the rôle of a parameter.

Polynomials ii

- ‘new wave’ polynomials à la Gambino-Kock [GK13]: define a category of polynomials having objects the diagrams

$$\mathfrak{f} : \; I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I$$

and suitable morphisms. To each such $\mathfrak{f}$ one can associate a polynomial endofunctor $P_{\mathfrak{f}}$ over $\mathcal{E}/I$, with $\mathfrak{f}$ as a parameter.

A few structural observations

Adjoints

Fibrations of (co)algebras tend to have reindexing preserving (co)limits; we can make this statement more precise by resorting to a well-known set of results about algebraic and presentable categories:

Theorem

Let $\mathcal{X}$ be κ-presentable and assume that the fibration of algebras is restricted to just the κ-accessible functors $\mathcal{X} \to \mathcal{X}$; then, each reindexing α^* has a left adjoint $\sum_\alpha$.

(dually for coalgebras)

Proof.
A clever application of a Freyd's swindle. □

Monadicity

Theorem

A fibration $\left[p \overset{\mathcal{E}}{\underset{\mathcal{A}}{\downarrow}} \right]$ is an EM-fibration if and only if there exists a morphism of fibrations $H : \left[p \overset{\mathcal{E}}{\underset{\mathcal{A}}{\downarrow}} \right] \to \left[\pi_{\mathcal{A}} \overset{\mathcal{X}\times\mathcal{A}}{\underset{\mathcal{A}}{\downarrow}} \right]$ which is monadic as a 1-cell in $\mathbf{Fib}/\mathcal{A}$.

This in turn is equivalent to the fact that H

- has a left adjoint fibered over $\mathcal{A}$;
- the Eilenberg-Moore object [Str72] for the monad HL induced by $L \dashv H$ is equivalent to p.

Proof.
Unwind the definition of monadic 1-cell in $\mathbf{Fib}/\mathcal{A}$. □

Monadicity

A more concrete reformulation of this criterion:

Fact

A fibration $\left[p \overset{\mathcal{E}}{\underset{\mathcal{A}}{\downarrow}} \right]$ is an EM-fibration iff it can be presented via a diagram of monadic categories

$$\hat{F} : \mathcal{A} \longrightarrow (\mathbf{Cat}/\mathcal{X})^{\mathrm{op}}_{m}$$

where on RHS there is the full subcategory of $\mathbf{Cat}/\mathcal{X}$ on monadic functors $U : \mathcal{A} \to \mathcal{X}$.

The link with graded monads

Graded monads [Smi08; MPS; FKM16; MU22; OWE20] are another way to consider "monads varying according to a parameter".

Fact

A monad in **Cat** is a lax functor $T : 1 \to \mathbf{Cat}$.

Definition

A graded monad is a lax functor $T : B\mathcal{M} \to \mathbf{Cat}$, where $\mathcal{M}$ is a monoidal category regarded as a one-object bicategory.

The object of algebras [Str72] for a graded monad T consists of its lax limit $\mathrm{llim}_{B\mathcal{M}} T \in \mathbf{Cat}$.

The link with graded monads

Our fibrations of EM-algebras can be seen as a generalization of graded monads:

- No monoidality for $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]_m$; we do not ask the domain of F to be a bicategory; a plain category will do;
- We do not consider algebras for all parameters at once (as in [MPS; DMS18]), but instead for each object separately.

Theorem

There is a bicategory $\Sigma\mathcal{A}$ such that

$$\mathbf{Cat}(\mathcal{A}, [\mathcal{X}, \mathcal{X}]_m) \cong \mathbf{Lax}(\Sigma\mathcal{A}, \mathbf{Cat})$$

and the diagram $\mathcal{A}^{\mathrm{op}} \to \mathbf{Cat} : A \mapsto \mathbf{Alg}(F_A)$ presents the lax limit of $\bar{F} : \Sigma\mathcal{A} \to \mathbf{Cat}$.

What now?

We really don’t know!

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