On the Fibration of Algebras

j/w Ahman, Coraglia, Castelnovo, Martins-Ferreira, Reimaa

Fosco Loregian

Tallinn University of Technology

XXVIII Incontro di Logica AILA

Plan d'œuvre

1. What is a 'fibration of algebras'
2. Motivating examples: simple types, polynomials
3. Exact sequences
4. Action of $\pi_1 B$ on zero sections

see arXiv:2408.16581 for more info

The subject of our study will be functors

$$F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$$

- $\mathcal{A}$: the category of parameters;
- $\mathcal{X}$: the category of carriers.

Equivalently,

$$F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$$

To each endofunctor $F_A = F(A, -)$ one can associate

1. the category of F-algebras $Alg(F_A)$
2. the category of F-coalgebras $coAlg(F_A)$
3. the category of EM-algebras $EM(F_A)$ if F_A is a monad
4. the category of coEM-algebras $coEM(F_A)$ if F_A is a comonad

Each of these associations defines a pseudofunctor (1. and 3. contravariant; 2. and 4. covariant)

$$\mathcal{A} \longrightarrow \mathbf{Cat}$$

as such (under the Grothendieck construction) a split fibration (1. and .3) or opfibration (2. and 4.)

Scope of this work:

- Study the op/fibrations

$$
\begin{array}{cccc}
\mathcal{A} \ltimes^{\mathit{Alg}} \mathcal{X} & \mathcal{A} \ltimes^{\mathit{coAlg}} \mathcal{X} & \mathcal{A} \ltimes^{\mathit{EM}} \mathcal{X} & \mathcal{A} \ltimes^{\mathit{coEM}} \mathcal{X} \\
\big\downarrow & \big\downarrow & \big\downarrow & \big\downarrow \\
\mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A}
\end{array}
$$

- Find examples
- Develop a general theory of these gadgets

Keywords:

representation theory, category of modules, categorical logic and type theory, graded monads, formal languages, coinduction for coalgebras, ...

A motivating example: the simple fibration.

Let $\mathcal{A}$ be a category with finite products; then there is a comonad

$$\mathcal{A} \to [\mathcal{A}, \mathcal{A}] : A \mapsto A \times -$$

the coKleisli category of which is the simple slice over A. It has

- the same objects of $\mathcal{A}$;
- morphisms $X \to Y$ are $\mathcal{A}(A \times X, Y)$

The Grothendieck construction gives rise to the simple fibration $\mathbf{s}\mathcal{A} \to \mathcal{A}$ over $\mathcal{A}$, where people in CLTT intepret simple type theories.

Similarly, one can collect the coEilenberg-Moore categories of $S_A = A \times -$; this gives the slice $\mathcal{A}/A$ as fiber (and the codomain $cod : \mathcal{A}^{\to} \to \mathcal{A}$ as associated opfibration).

Other examples come from the theory of automata / transition systems:

- Mealy-type automata are coalgebras for the parametric functor $R_{AB} = (A \times -)^B$ (parametric in input and output);
- labeled transition systems are coalgebras for $Q_A = 2^{A\times -}$ (parametric in the set of labels);

In the first case, one can define an endofunctor $\hat{R} : (A, B, X) \mapsto R_{AB}X$,

$$\begin{array}{ccc} Set^{op} \times Set \times Set & \xrightarrow{\hat{R}} & Set^{op} \times Set \times Set \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & Set^{op} \times Set & \end{array}$$

so that $Set \ltimes_R^{coAlg} Set$ is just $coAlg(\hat{R})$.

Similarly in the second case.

Another example is given by polynomial functors: define the category **pol** having

- objects the sequences $\mathfrak{p} : I \leftarrow B \to A \to I$;
- morphisms suitable cartesian squares $(B \to A) \to (B' \to A')$.

Gambino and Kock define polynomial functors $P_\mathfrak{p} : \mathcal{E} \to \mathcal{E}$ associated to each polynomial shape $\mathfrak{p}$ in a way that $\mathfrak{p} \mapsto Alg(P_\mathfrak{p})$ is a pseudofunctor $\mathbf{pol}^{op} \to \mathbf{Cat}$.

The category **pol** $\ltimes_P^{Alg} \mathcal{E}$ is the fibration of polynomials.

Gambino and Hyland: «assume an initial algebra for polynomials in one variable exists, then one exists for polynomials in all variables.»

Theorem

[GH] $\iff$ Initial objects are created by reindexing functors in **pol** $\ltimes_P^{Alg} \mathcal{E}$.

These examples motivate a

Guiding principle

A certain property of the diagram

$$\mathcal{A} \xleftarrow{\text{carrier}} \mathcal{A}^{\to} \xrightarrow{\text{opfib}} \mathcal{A}$$

can likely be better understood when generalized to a property of

$$\mathcal{X} \xleftarrow{i} \mathcal{A} \ltimes^{\text{coEM}} \mathcal{X} \xrightarrow{p} \mathcal{A}.$$

Let $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ be a parametric monad. (All will dualize to comonads)

Limits in $\mathcal{A} \ltimes^{EM} \mathcal{X}$ are computed in a particularly straightforward way, created by the forgetful functor

$$\langle p, i \rangle : \mathcal{A} \ltimes^{EM} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}. \qquad (\heartsuit)$$

In fact,

Theorem

The functor $\langle p, i \rangle$ is monadic.

More is true:

Theorem

$\mathcal{A} \ltimes^{EM} \mathcal{X}$ is the Eilenberg-Moore category of a monad

$$\hat{T} : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$$

fibered over the projection $\pi_{\mathcal{A}} : \mathcal{A} \times \mathcal{X} \to \mathcal{A}$. $\hat{T}(A, X) = (A, T_A X)$

Let's study more properties of the pair $\langle p, i\rangle$ in $(\heartsuit)$. Let again T be a parametric monad

Assume

- that $\mathcal{X}$ has a terminal object;
- that $\mathcal{A}$ has an initial object.

$$\mathcal{X} \underset{V}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{A} \ltimes^{EM} \mathcal{X} \underset{!}{\overset{p^T}{\underset{\longleftarrow}{\overset{\twoheadrightarrow}{\perp}}}} \mathcal{A}$$

- V has a left adjoint $(\Phi X :=$ free $T_\varnothing$-algebra; in particular $\Phi X := X$ iff $T_\varnothing \cong id)$
- p^T has a right adjoint $^!(-)$ $(^!A :=$ terminal object in the fiber/$A)$;

Theorem

Let $\mathcal{A}, \mathcal{X}$ be pointed categories; then there is an exact sequence of left adjoints

$$1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \ltimes^{EM} \mathcal{X} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A} \longrightarrow 1$$

This POV goes pretty far:

Proposition

Let $\mathcal{A}, \mathcal{X}$ be pointed categories.
There is a 2-category $\mathbf{Seq}^1(\mathcal{A}, \mathcal{X})$ of sequences of $\mathcal{A}$ by $\mathcal{X}$

$$1 \longrightarrow \mathcal{X} \xrightarrow{i} \mathcal{E} \overset{p}{\twoheadrightarrow} \mathcal{A} \longrightarrow 1.$$

where p is a fibration.

There is a subcategory $\mathbf{Ext}^1(\mathcal{A}, \mathcal{X})$ spanned by the objects of $\mathbf{Seq}^1(\mathcal{A}, \mathcal{X})$ such that $p \circ i$ is constant at the zero object.

Theorem (teaser)

$\mathbf{Ext}^1(\mathcal{A}, \mathcal{X})$ is a symmetric monoidal category.

Recall that any fibration of spaces $p : E \to B$ renders $E_b = p^{-1}(b)$ a $\pi_1(B)$-space; there is an analogue of this result here.

Theorem

Let $\mathcal{A}$ have an initial object, $p : \mathcal{E} \to \mathcal{A}$ be a fibration such that

[it doesn't matter; we call them pruned fibrations]

then there is a canonical way to build a parametric monad $T_p : \mathcal{A} \to [\mathcal{E}_\varnothing, \mathcal{E}_\varnothing]$ so that the base of p acts over the fiber of p on the initial object. Moreover, T_p is a monad such that

[it doesn't matter; we call them pruned monads].

This is part of a reflection

$$\mathbf{Fib}/\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Mnd}_\varnothing(\mathcal{A})$$

identifying pruned monads with pruned fibrations.

There are

- more examples in representation theory, algebraic topology, categorical algebra... (ideally, there are 'as many examples as there are endofunctors')
- more theorems about the 2-category of extensions, that mimick the 'hands-on' theory of *Ext* groups;
- consequences of monadicity: $\mathcal{A} \ltimes^{EM} \mathcal{X}$ is a 'rigid' object whose nature is specified by the formal theory of monads in $\mathit{Fib}/\mathcal{A}$
- generalizations to a theory of 1-cells of $\mathit{Fib}/\mathcal{A}$ monadic over a (bi)fibration other than the trivial one...

It's a long-term project.