# On the Fibration of Algebras

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- 1 What is a 'fibration of algebras'
- 2 Motivating examples: simple types, polynomials
- 3 Exact sequences
- **4** Action of  $\pi_1 B$  on zero sections

# see 🖺 arXiv:2408.16581 for more info

The subject of our study will be functors

$$F: \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$$

- A: the category of parameters;
- X: the category of carriers.

Equivalently,

$$F: \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$$

To each endofunctor  $F_A = F(A, -)$  one can associate

- 1. the category of F-algebras  $Alg(F_A)$
- 2. the category of F-coalgebras  $coAlg(F_A)$
- 3. the category of EM-algebras  $EM(F_A)$  if  $F_A$  is a monad
- 4. the category of coEM-algebras  $coEM(F_A)$  if  $F_A$  is a comonad

Each of these associations defines a pseudofunctor (1. and 3. contravariant; 2. and 4. covariant)

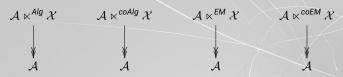
$$A \longrightarrow Cat$$

as such (under the Grothendieck construction) a split fibration (1. and .3) or opfibration (2. and 4.)

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## Scope of this work:

Study the op/fibrations



- Find examples
- · Develop a general theory of these gadgets

## Keywords:

representation theory, category of modules, categorical logic and type theory, graded monads, formal languages, coinduction for coalgebras, ...

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A motivating example: the simple fibration.

Let  $\ensuremath{\mathcal{A}}$  be a category with finite products; then there is a comonad

$$\mathcal{A} \rightarrow [\mathcal{A}, \mathcal{A}] : A \mapsto A \times -$$

the coKleisli category of which is the simple slice over A. It has

- the same objects of A;
- morphisms  $X \to Y$  are  $\mathcal{A}(A \times X, Y)$

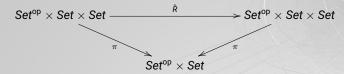
The Grothendieck construction gives rise to the simple fibration  $\mathbf{s}\mathcal{A}\to\mathcal{A}$  over  $\mathcal{A}$ , where people in CLTT interpret simple type theories.

Similarly, one can collect the coEilenberg-Moore categories of  $S_A = A \times -$ ; this gives the slice  $\mathcal{A}/A$  as fiber (and the codomain  $cod : \mathcal{A}^{\to} \to \mathcal{A}$  as associated opfibration).

Other examples come from the theory of automata / transition systems:

- Mealy-type automata are coalgebras for the parametric functor  $R_{AB} = (A \times -)^B$  (parametric in input and output);
- labeled transition systems are coalgebras for  $Q_A = 2^{A \times -}$  (parametric in the set of labels);

In the first case, one can define an endofunctor  $\hat{R}:(A,B,X)\mapsto R_{AB}X$ ,



so that  $Set \ltimes_{R}^{coAlg} Set$  is just  $coAlg(\hat{R})$ . Similarly in the second case.

Another example is given by polynomial functors: define the category pol having

- objects the sequences  $\mathfrak{p}:I\leftarrow B\rightarrow A\rightarrow I$ ;
- morphisms suitable cartesian squares  $(B \to A) \to (B' \to A')$ .

Gambino and Kock define polynomial functors  $P_{\mathfrak{p}}: \mathcal{E} \to \mathcal{E}$  associated to each polynomial shape  $\mathfrak{p}$  in a way that  $\mathfrak{p} \mapsto Alg(P_{\mathfrak{p}})$  is a pseudofunctor **pol**<sup>op</sup>  $\to$  **Cat**.

The category **pol**  $\ltimes_P^{Alg} \mathcal{E}$  is the fibration of polynomials.

Gambino and Hyland: «assume an initial algebra for polynomials in one variable exists, then one exists for polynomials in all variables.»

## Theorem

[GH]  $\iff$  Initial objects are created by reindexing functors in **pol**  $\ltimes_{\mathcal{D}}^{Alg} \mathcal{E}$ .

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## These examples motivate a

## **Guiding principle**

A certain property of the diagram

$$\mathcal{A} \stackrel{\mathsf{carrier}}{\longleftarrow} \mathcal{A}^{\rightarrow} \stackrel{\mathsf{opfib}}{\longrightarrow} \mathcal{A}$$

can likely be better understood when generalized to a property of

$$\mathcal{X} \longleftarrow {}^{i} \mathcal{A} \ltimes^{\mathsf{coEM}} \mathcal{X} \stackrel{\mathsf{p}}{\longrightarrow} \mathcal{A}.$$

Let  $T: A \times X \to X$  be a parametric monad. (All will dualize to comonads)

Limits in  $\mathcal{A}\ltimes^{\textit{EM}}\mathcal{X}$  are computed in a particularly straightforward way, created by the forgetful functor

$$\langle p, i \rangle : \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}.$$
 ( $\heartsuit$ )

In fact,

### **Theorem**

The functor  $\langle p, i \rangle$  is monadic.

More is true:

#### **Theorem**

 $\mathcal{A} \ltimes^{\mathit{EM}} \mathcal{X}$  is the Eilenberg-Moore category of a monad

$$\hat{T}: \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$$

fibered over the projection  $\pi_A : A \times \mathcal{X} \to A$ .  $\hat{T}(A, X) = (A, T_A X)$ 

Let's study more properties of the pair  $\langle p,i\rangle$  in  $(\heartsuit)$ . Let again T be a parametric monad

#### Assume

- that X has a terminal object;
- that A has an initial object.

$$\mathcal{X} \xrightarrow[V]{\Phi} \mathcal{A} \ltimes^{EM} \mathcal{X} \xrightarrow[V]{\rho^{T}} \mathcal{A}$$

- V has a left adjoint  $(\Phi X := \text{free } T_{\varnothing}\text{-algebra}; \text{ in particular } \Phi X := X \text{ iff } T_{\varnothing} \cong id)$
- $p^T$  has a right adjoint !(-) (!A := terminal object in the fiber/A);

## Theorem

Let  $\mathcal{A}, \mathcal{X}$  be pointed categories; then there is an exact sequence of left adjoints

$$1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \ltimes^{EM} \mathcal{X} \longrightarrow \mathcal{A} \longrightarrow 1$$

## This POV goes pretty far:

## Proposition

Let A,  $\mathcal{X}$  be pointed categories.

There is a 2-category **Seq**<sup>1</sup>( $\mathcal{A}, \mathcal{X}$ ) of sequences of  $\mathcal{A}$  by  $\mathcal{X}$ 

$$1 \longrightarrow \mathcal{X} \stackrel{i}{\longrightarrow} \mathcal{E} \stackrel{p}{\longrightarrow} \mathcal{A} \longrightarrow 1.$$

where p is a fibration.

There is a subcategory  $\operatorname{Ext}^1(\mathcal{A},\mathcal{X})$  spanned by the objects of  $\operatorname{Seq}^1(\mathcal{A},\mathcal{X})$  such that  $p \circ i$  is constant at the zero object.

## Theorem (teaser)

 $\operatorname{Ext}^1(\mathcal{A},\mathcal{X})$  is a symmetric monoidal category.

Recall that any fibration of spaces  $p: E \to B$  renders  $E_b = p^{-1}(b)$  a  $\pi_1(B)$ -space; there is an analogue of this result here.

#### **Theorem**

Let A have an initial object,  $p: \mathcal{E} \to A$  be a fibration such that

[it doesn't matter; we call them pruned fibrations]

then there is a canonical way to build a parametric monad  $T_p: \mathcal{A} \to [\mathcal{E}_\varnothing, \mathcal{E}_\varnothing]$  so that the base of p acts over the fiber of p on the initial object. Moreover,  $T_p$  is a monad such that

[it doesn't matter; we call them pruned monads].

This is part of a reflection

$$\mathsf{Fib}/\mathcal{A} \xrightarrow{\hspace{1cm} \bot \hspace{1cm}} \mathsf{Mnd}_\varnothing(\mathcal{A})$$

identifying pruned monads with pruned fibrations.

#### There are

- more examples in representation theory, algebraic topology, categorical algebra... (ideally, there are 'as many examples as there are endofunctors')
- more theorems about the 2-category of extensions, that mimick the 'hands-on' theory of Ext groups;
- consequences of monadicity:  $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$  is a 'rigid' object whose nature is specified by the formal theory of monads in  $\mathsf{Fib}/\mathcal{A}$
- generalizations to a theory of 1-cells of Fib/A monadic over a (bi)fibration other than the trivial one...

It's a long-term project.

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