The art of $\int$ – notable integrals in Category Theory

Fosco Loregian TALTECH

December 19, 2019

ItaCa Liber I

Plan de l'œuvre

«*I have always disliked analysis*»

P.J. Freyd (Algebraic real analysis)

Aims:

- perform integrals (=co/ends) in category theory;
- coincidence between integration and co/ends: accidental?
- forget about analysis and do category theory. Twofold aim:
    - categorify convolution structures, distributions, Fourier theory, power series...
    - describe constructions (like Stokes' theorem) using coends.

Caveat: this wants to be a "light" talk (and partly self-promotion).

Coends

Co/ends

Coends are universal objects associated to functors

$$T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

Defined as

- $\int_C T(C,C) \longrightarrow \prod_{C\in\mathcal{C}} T(C,C) \rightrightarrows \prod_{C\to C'} T(C,C')$
- $\coprod_{C\to C'} T(C,C') \rightrightarrows \coprod_{C\in\mathcal{C}} T(C,C) \longrightarrow \int^C T(C,C)$
- The end $\int_C T$ = object of invariants for the action of T on arrows;
- The coend $\int^C T$ = orbit space for the action of T on arrows.

$$\int_C T \xrightarrow{\textit{terminal}} T(X,X) \qquad T(X,X) \xrightarrow[\textit{initial}]{} \int^C T$$

Co/ends

Examples:

- $F, G : \mathcal{C} \to \mathcal{D}$ functors: then

$$\mathsf{Nat}(F, G) \cong \int_C \mathcal{D}(FC, GC)$$

- A, B two G-modules: then

$$A \otimes_G B \cong \operatorname{colim}\left(\bigoplus_{g \in G} A \otimes B \overset{g \otimes 1}{\underset{1 \otimes g}{\rightrightarrows}} A \otimes B \right)$$

- A, B two G-modules: then

$$\hom_G(A, B) \cong \lim\left(\hom(A, B) \rightrightarrows \prod_{g \in G} \hom(A, B) \right)$$

Co/ends

Why integrals?

- They depend contra-covariantly from their domain:

$$\int_X f(x)dx$$

- They satisfy a Fubini rule

$$\begin{aligned}\int^{(C,D)} T(C,D,C,D) &\cong \int^D \int^C T(C,C,D,D)\\ &\cong \int^C \int^D T(C,C,D,D)\end{aligned}$$

- They provide analogues for a theory of integrations

Coend calculus

Coends abound in Mathematics:

- Yoneda lemma $\int_C [\mathsf{y}C(X), FX] \cong FC$
- Kan extensions $\int^A \hom(GA, -) \times FA \cong \mathsf{Lan}_G F$
- Nerves and realisations $\mathsf{Lan}_{\mathsf{y}} F \dashv \mathsf{Lan}_F \mathsf{y}$
- Weighted co/limits $\mathsf{colim}^W F \cong \int^A WA \otimes FA$
- Profunctors
- Operads
- Functional programming
- …

Analysis

Dirac deltas

Let $\mathsf{y} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding.

Every object of the form $\mathsf{y}C$ is tiny, so it is a functor concentrated on the “point” $C \in \mathcal{C}$. Yoneda lemma says that

$$\int_X \hom(\mathsf{y}C(X), FX) \cong FC$$

(“C-points” of a presheaf F = elements of FC)

Or dually,

$$\int^C \mathsf{y}C(X) \times FX = \mathsf{y}C \otimes_{\mathcal{C}} F \cong FC$$

(the Dirac δ functor concentrated on $C \in \mathcal{C}$ evaluates functors on points).

Stokes' theorem

Fix a manifold X.

- Let $\Omega : \mathbf{N} \to \mathsf{Mod}(\mathbf{R})$ the functor sending n to the set of differential n-forms $\Omega^n(X)$ and

$$d_n : \Omega^n(X) \to \Omega^{n+1}(X)$$

  the exterior derivative.
- Let $C : \mathbf{N}^{\mathrm{op}} \to \mathsf{Mod}(\mathbf{R})$ the functor sending n to (the vector space over) smooth maps $Y \to X$ where Y is closed n-dimensional oriented manifold: $\partial : C_{n+1} \to C_n$ is the geometric boundary.

Stokes' theorem

$$C \otimes \Omega : \mathbf{N}^{\mathsf{op}} \times \mathbf{N} \to \mathsf{Vect}$$

is a functor. There is a map

$$\textstyle\int : C \otimes \Omega \to \mathbf{R}$$

$$(Y \xrightarrow{\varphi} X, \omega) \mapsto \textstyle\int_Y \varphi^* \omega$$

Theorem (Stokes): The square

$$\begin{array}{ccc} C_{n+1} \otimes \Omega_n & \xrightarrow{\partial \otimes 1} & C_n \otimes \Omega_n \\ \downarrow{\scriptstyle 1 \otimes d} & & \downarrow{\scriptstyle \int_n} \\ C_{n+1} \otimes \Omega_{n+1} & \xrightarrow[\int_{n+1}]{} & \mathbf{R} \end{array}$$

is commutative for every $n \in \mathbf{N}$. $\int^n C \otimes \Omega$ is a certain H^0...

Distributions

Let **Prof** be the bicategory of profunctors:

- objects: categories
- 1-cells $p : \mathcal{C} \rightsquigarrow \mathcal{D}$: functors $p : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to$ **Set**
- 2-cells $\alpha : P \Rightarrow q$ natural transformations.

A profunctor is also called a distributor.

A Lawvere distribution is a left adjoint between two toposes;

dist. between sheaves on $\mathcal{C}, \mathcal{D}$
$\|$
profunctors $p : \mathcal{C} \rightsquigarrow \mathcal{D}$

(Dirac distributions over a topos $\mathcal{E}$ are points of that topos (geometric morphisms $p : \mathcal{E} \to$ **Set**); complies with intuition)

Convolutions

Let G be a group; the set of regular functions $f : G \to \mathbf{R}$ has a convolution operation

$$f * g = y \mapsto \int_G fx \cdot g(y - x) d\mu_G$$

Let $\mathcal{C}$ be a monoidal category; the category $[\mathcal{C}, \textbf{Set}]$ becomes a monoidal category with a convolution operation of presheaves:

$$F * G = C \mapsto \int^{XY} FX \times GY \times \mathcal{C}(X \otimes Y, C)$$

(what if $\mathcal{C}$ is closed? You recover the above formula)

Fourier theory

Fact:

$$\textbf{Prof}(\mathcal{C}, \mathcal{D}) \cong \mathsf{LAdj}([\mathcal{C}^{\mathrm{op}}, \textbf{Set}], [\mathcal{D}^{\mathrm{op}}, \textbf{Set}])$$

Let us now replace **Set** with a *-autonomous category $\mathcal{V}$.

A profunctor $K : \mathcal{C} \rightsquigarrow \mathcal{D}$ between monoidal categories is a multiplicative kernel if the associated

$$\hat{K} : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$$

is a strong monoidal adjunction wrt convolution product.

Fourier theory

The K-Fourier transform $f \mapsto \mathfrak{F}_K(f) : \mathcal{D} \to \mathcal{V}$, obtained as the image of $f : \mathcal{C} \to \mathcal{V}$ under the left Kan extension $\mathrm{Lan}_y K : [\mathcal{C}, \mathcal{V}] \to [\mathcal{D}, \mathcal{V}]$.

$$\mathfrak{F}_K(f) : X \mapsto \int^A K(A, X) \otimes fA.$$

The dual Fourier transform is defined as:

$$\mathfrak{F}^\vee(g) : Y \mapsto \int_A [K(A, X), gA]$$

(prove the relation $\mathfrak{F}_K^\vee(g) \cong \mathfrak{F}_K(g^*)^*$)

$\mathfrak{F}_\mathrm{y}$ is the identity functor; analogue in analysis, what is the Fourier transform of δ?

Fourier theory

- $\mathfrak{F}_K$ preserves the upper convolution of presheaves f, g, defined as

$$f \mathbin{\overline{*}} g = \int^{AA'} fA \otimes gA' \otimes \mathcal{C}(A \otimes A', -);$$

dually,

- $\mathfrak{F}_K^\vee$ preserves the lower convolution of presheaves f, g, defined as

$$f \mathbin{\underline{*}} g = \int_{AA'} (fA^* \otimes (gA')^* \otimes \mathcal{C}(A \otimes A', -))^*$$

Fourier theory

Define the pairing $(\mathcal{C}, \mathcal{V}) \times (\mathcal{C}, \mathcal{V}) \to \mathcal{V}$ as the twisted form of functor tensor product

$$\langle f, g \rangle = \int^{A} fA^{*} \otimes gA$$

If K is a kernel such that $\mathrm{Lan}_y K$ is fully faithful, we have Parseval formula:

$$\langle f, g \rangle \cong \langle \mathfrak{F}_K(f), \mathfrak{F}_K(g) \rangle.$$

Bibliography

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- Day, Brian J. Monoidal functor categories and graphic Fourier transforms. Theory and Applications of Categories 25.5 (2011): 118-141.

When you come across a paper with page after page of nothing but enriched categories and coend formulas: