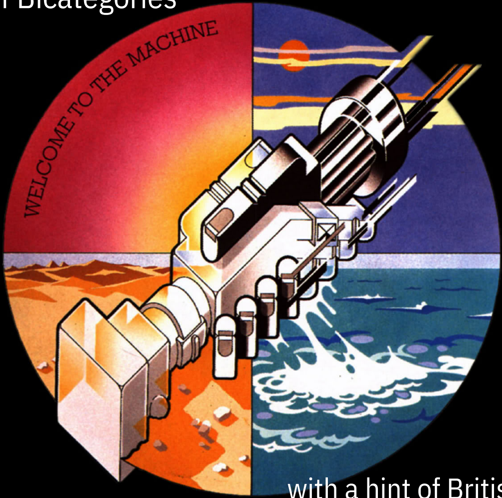
A Tale of Bicategories



with a hint of British prog

Prolegomena

j/w G. Boccali, A. Laretto, S. Luneia; EPTCS.397.1 .

Actually, there is more to this story:

- Boccali, G., Laretto, A., ____, & Luneia, S. “Completeness for categories of generalized automata.” LIPIcs.CALCO.2023.20 ;
- Boccali, G., Femić, B., Laretto, A., ____, & Luneia, S. “The semibicategory of Moore automata.” arXiv:2305.00272 .

A theory of abstract automata

The accent is on 'abstract'

Let **K** be a strict 2-category with all finite weighted limits.

Fix a 0-cell C, an endo-1-cell $f : C \to C$ and consider as building blocks of our theory

- the inserter $u : I(f, 1_C) \to C$ or 'object of algebras' for f;
- for every $b : B \to C$ the comma object C/b (equipped with its canonical projection $C/b \to C$);
- the comma object $(f/b) \to C$.

The accent is on ‘abstract’

Then, the object of (f, b)-Mealy machines is the pullback on the left of

$$\begin{array}{ccc} \mathbf{Mly}(f,b) & \longrightarrow & (f/b) \\ \downarrow & \lrcorner & \downarrow \\ I(f,1_C) & \longrightarrow & C \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{Mre}(f,b) & \longrightarrow & C/b \\ \downarrow & \lrcorner & \downarrow \\ I(f,1_C) & \longrightarrow & C \end{array}$$

and the object of (f, b)-Moore machines is the pullback on the right.

As such, **Mly** and **Mre** are parametric functors of type

$$\mathbf{K}(C,C)^{\mathrm{op}} \times \mathbf{K}/C \longrightarrow \mathbf{K}/C$$

The accent is on 'abstract' (but let's do it in Cat)

If $\mathbf{K} = \mathbf{Cat}$ and $b : 1 \to C$ is a single object, these definitions specialize to

- the category of Mealy automata, where objects and morphisms are of the form

$$
\begin{array}{ccccc}
e & \xleftarrow{\;d\;} & Fe & \xrightarrow{\;s\;} & b \\
{\scriptstyle \varphi}\,\downarrow & & \downarrow\,{\scriptstyle F\varphi} & & \| \\
e' & \xleftarrow{\;d\;} & Fe & \xrightarrow{\;s\;} & b
\end{array}
$$

- the category of Moore automata, where objects and morphisms are of the form

$$
\begin{array}{ccccc}
e & \xleftarrow{\;d\;} & Fe, e & \xrightarrow{\;s\;} & b \\
{\scriptstyle \varphi}\,\downarrow & & \downarrow \quad \downarrow & & \| \\
e' & \xleftarrow{\;d\;} & Fe', e' & \xrightarrow{\;s\;} & b
\end{array}
$$

The accent is on ‘abstract’ (but let’s do it in Cat)

In particular, if $F_A : \mathbf{K} \to \mathbf{K}$ is the functor depending on an object A (an ‘Alphabet’) Mealy and Moore automata are respectively diagrams of the form (E, d, s):

$$E \xleftarrow{d} A \otimes E \xrightarrow{s} B$$

and of the form

$$E \xleftarrow{d} A \otimes E, E \xrightarrow{s} B$$

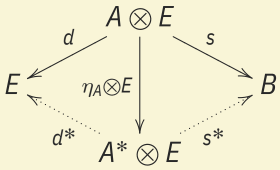
This is (a flavour of) what people usually call ‘Mealy’ and ‘Moore’ automata.

- $d : A \otimes E \to E$ is an action of A on E (a dynamical system);
- s is an output function (think of $B = \{0, 1\}$ or $B = [0, 1]$, etc.)

The accent is on 'abstract' (but let's do it in Cat)

If **K** is monoidal closed, $F_A = A \otimes -$ is colimit-preserving and its algebras coincide with the coalgebras of its right adjoint $[A, -]$. This allows a number of deductions:

- If **K** has countable sums, $d : A \otimes E \to E$ is an action of $A^* := \sum_n A^n$, and s extends similarly:



This is called the canonical extension of (E, d, s).

The accent is on ‘abstract’ (but let’s do it in Cat)

- If $(\mathbf{K}, \otimes)$ is monoidal and $T : \mathbf{K} \to \mathbf{K}$ is a commutative monad over it, we can lift the monoidal structure of **K** making the free functor $F : \mathbf{K} \to \mathbf{Kl}(T)$ strong monoidal.

  Machines in $\mathbf{Kl}(T)$ are non-deterministic versions of the ones in **K**.

  Take T the powerset monad on **Set**, or a distribution/probability monad like the one of finite distributions –whose algebras are convex sets, and free algebras affine simplices).

The accent is on ‘abstract’ (but let’s do it in Cat)

- If **K** is cocomplete (e.g., locally presentable), so are **Mly**(A, B), **Mre**(A, B) for every A, B –with colimits created by the forgetful into $\mathcal{K}$ and connected limits created by the functor in the commæ.

In particular, the terminal objects of **Mly**(A, B), **Mre**(A, B) are respectively

$$[A^+, B] \qquad\qquad [A^*, B]$$

(A^+ = free semigroup on A; A^* = free monoid on A).

Observe that this can be deduced from the fact that when **K** is closed, we can characterize automata coalgebraically, see some work of Jacobs.

(Semi)bicategories of automata

A tale of bicategories

When **K** is Cartesian, $\mathbf{Mly}(A, B)$ is the hom-category of a bicategory.[1]

The slickest way to see this is the following:

- consider the monoidal category **K** as a bicategory $\Sigma\mathbf{K}$ with a single object;
- define the bicategory $C(\mathbf{K})$ as the bicategory $\mathbf{Psd}(\mathbf{N}, \Sigma\mathbf{K})$ of pseudofunctors and lax natural transformations. Then, a 1-cell in $C(\mathbf{K})$ consists of a pair $(E, x) : E \otimes A \xrightarrow{x} B \otimes E$.

Evidently, if $(\mathbf{K}, \otimes)$ is Cartesian, the universal property of products splits every x as $\langle s, d \rangle$ where (d, s) fit in the previous span.

Clearly, $C(\mathbf{K})$ exists for every monoidal category!

[1] This motivates the compact notation $(E, x) : A \twoheadrightarrow B$ to refer to a Mealy machine valued in **K**.

A hint of prog

The situation is not as straightforward for Moore automata, as there are no identity 1-cells.

We investigate the situation in arXiv:2305.00272 outlining that

- A semibicategory is ‘like a bicategory, but without identity 1-cells’
- There exists a semibicategory **Mre** of Moore-type automata, a functor

$$J : \mathbf{Mre} \longrightarrow \mathbf{Mly}$$

and a right adjoint $\mathbf{Mly}(A, B) \to \mathbf{Mre}(A, B)$, altogether forming the components of a local adjunction.

A la façon de Guitart

In 1974, R. Guitart produced a span representation for Mealy automata:

The category $\mathbf{Mac}^s$ is the sub-bicategory of $\mathbf{Span}(\mathbf{Cat})$ where the left leg is a discrete opfibration.

There is a strict equivalence of bicategories between

- the 1- and 2-full sub-bicategory of $\mathbf{Mac}^s$ spanned by monoids (=one-object categories);
- the 2-full sub-bicategory of **Mly** (over **Set**) whose 1-cells are Mealy automata between monoids such that the representation of A^* on E in $E \xleftarrow{d^*} A^* \otimes E \xrightarrow{s^*} B$ induces a functor $\Sigma : \mathcal{E}[d^*] \to B$, when B is a monoid.

Machines valued in a bicategory

You can't just '*is just*' me and expect me to believe you

> A monoidal category *is just*™ a bicategory with a single object.

But then, do the definition given above make sense when instead of **K** we consider a bicategory $\mathbb{B}$ with more than one object?

This idea is not *entirely* new; it resembles old (and obscure) work of Bainbridge, modeling the state space of abstract machines as a functor, of which one can take the left/right Kan extension along an 'input scheme'. See work of Petrişan et al.

A bimachine is a span in...

Definition

Let $\mathbb{B}$ be a bicategory; a bicategorical Moore (biMoore) machine in $\mathbb{B}$ is a diagram of 2-cells

$$e \Longleftarrow e \circ i, e \Longrightarrow o$$

between 1-cells e, i, o.[2]

The fact that this span exists, *coherces the types* of i, o, e in such a way that i must be an endomorphism of an object A.

$$A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A \xrightarrow{i} A, \dots$$

all make sense.

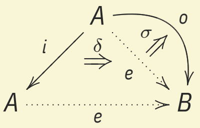
In the monoidal case, the fact that an input 1-cell stands on a different level from an output was completely obscured by the fact that every 1-cell is an endomorphism.

[2] A 1-cell of states (états), of inputs, and of outputs.

Everything will be made a Kan extension

Recall that the terminal objects of $\mathbf{Mly}(A, B)$, $\mathbf{Mre}(A, B)$ are respectively $[A^+, B]$, $[A^*, B]$.

Analogously, given that a biMoore of fixed input and output i, o consists of a way of filling the dotted arrows in



with 1- and 2-cells, we have

> The terminal object of the category of biMoore machines[3] is the right extension of $o : A \to B$ along the free monad $i^\sharp : A \to A$.

[3] With the obvious choice of morphisms, *mutatis mutandis*.

Examples

biMoore in Cat

Regarding **Cat** as a strict 2-category, a biMoore machine is a functor $E : \mathcal{C} \to \mathcal{D}$ closing a span $\mathcal{C} \xleftarrow{I} \mathcal{C} \xrightarrow{O} \mathcal{D}$ with suitable 2-cells.

If $\mathcal{D} = \mathbf{Set}$, states and output are presheaves, and E is acted by an endofunctor; in this case, the behaviour of the terminal machine can be described as a known object: unpacking the end that defined $Ran_{I^\natural} O$ we obtain the functor

$$A \longmapsto [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, I^\natural \_), O)$$

sending an object A to the set of natural transformations $\alpha : \mathcal{C}(A, I^\natural \_) \Rightarrow O$; to each generalised A-element of $I^\natural C$ corresponds an element of the output space $\Upsilon_C(u) \in OC$.

biMoore in Prof

In the bicategory **Prof** of profunctors, a biMoore machine $E : I \rightharpoonup O$ consists of a digraph I of inputs, and parallel profunctors E, O of states and output.

In the special case of $\{0, 1\}$-enriched profunctors (i.e., relations), the Kan extension of behaviour reduces to the maximal E such that $E \subseteq O$ and $E \circ I^\natural \subseteq E$ (here $\circ$ is the relational composition). So $R = \mathrm{Ran}_{I^\natural} O$ is the relation defined as

$$(a, b) \in R \iff \forall a' \in A.((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

This relation expresses *reachability* of b from a:

$$a\,R\,b \iff \Big((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a'\,O\,b\Big)$$

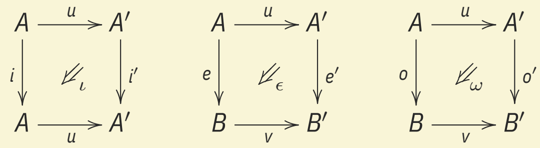
New maps

Intertwiners

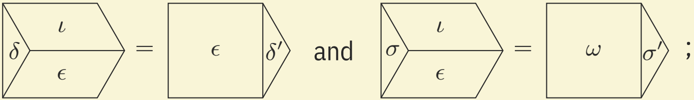
Definition (Intertwiner between bicategorical machines)

Consider two bicategorical Mealy machines $(e, \delta, \sigma)_{A,B}, (e', \delta', \sigma')_{A',B'}$ on different bases.

An *intertwiner* $(u, v) : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ consists of a pair of 1-cells $u : A \to A', v : B \to B'$ and a triple of 2-cells ι, ϵ, ω disposed as



such that



Intertwiners

When it is spelled out in the case when $\mathbb{B}$ has a single 0-cell, this notion does not reduce to any previously known one.

An intertwiner between (monoidal) machines $(E, d, s)_{I,O}$ and $(E', d', s')_{I',O'}$ consists of a pair of objects $U, V \in \mathcal{K}$, such that

1. there exist morphisms $\iota : I' \otimes U \to V \otimes I, \epsilon : E' \otimes U \to V \otimes E, \omega : O' \otimes U \to V \otimes O$;
2. the following two identities hold:

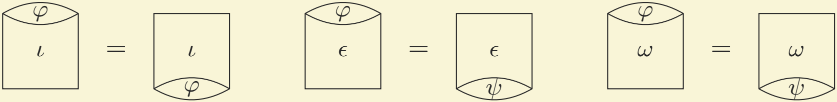
$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$
$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

Intertwiner 2-cells

Intertwiners between machines support a notion of higher morphisms:

Definition (2-cell between machines)

Let $(u, v), (u', v') : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ be two parallel intertwiners; a 2-cell $(\varphi, \psi) : (u, v) \Rightarrow (u', v')$ consists of a pair of 2-cells $\varphi : u \Rightarrow u'$, $\psi : v \Rightarrow v'$ such that



This notion is *not* trivial in the monoidal case!

Vistas

Monoidal topology and automata

Let $T : \mathbf{Set} \to \mathbf{Set}$ be a monad, and $\mathcal{V}$ a quantale.

Tholen, Clementino et al. build locally thin bicategories of $(T, \mathcal{V})$-matrices and $(T, \mathcal{V})$-categories providing a unified description of the categories of topological spaces, approach spaces, metric and ultrametric, probabilistic-metric closure spaces...

BiMoore and biMealy machines, when instantiated in $(T, \mathcal{V})$-**Prof**, a 2-categorical way to look at topological, (ultra)metric ways to study behaviour of a state machine The reachability relation becomes topological, (ultra)metric, probabilistic, sequential... according to suitable choices of T and $\mathcal{V}$.

Rabin-Scott, and profunctors

Nondeterminism via Kleisli construction is a powerful tool.

If automata in the Kleisli category of the powerset monad are nondeterministic automata in Set, biMoore/biMealy in **Prof** must be nondeterministic.

Conjecture

One can address nondeterministic biMoore automata in $\mathbb{B}$ as deterministic bicategorical automata in a proarrow equipment, porting all the paraphernalia (minimisation, behaviour, and bisimulation) into a bigger conceptual framework.

The En(i)d

The En¡d is a simphonic prog rock band from Southampthon;
suggested listening: *Ærie Færie Nonsense* and *Trippin the Light Fantastic*.