OF LIMS AND SETS

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1. Reduction of limits to products and equalizers

Definition 1.1 (Some terminology)**.** A (small) *diagram* in a category $\mathcal{C}$ is a functor

$$\mathcal{J} \longrightarrow \mathcal{C} \tag{1.1}$$

whose domain is a small category. A *cone* for a diagram D consists of a pair (X, c) where X is an object of $\mathcal{C}$ called *tip* of the cone, and c is a natural transformation $\Delta X \Rightarrow D$ from the constant functor at X; so c consists of a family of arrows in $\mathcal{C}$,

$$c_j : X \longrightarrow Dj \tag{1.2}$$

the *components* of the cone, such that for every morphism $f : i \to j$ in $\mathcal{J}$ the triangle

$$\begin{array}{ccc} & X & \\ {}^{c_i}\swarrow & & \searrow^{c_j} \\ Di & \xrightarrow[Df]{} & Dj \end{array} \tag{1.3}$$

is commutative. The category of cones for D has

- objects the cones (X, c) for D;
- arrows $h : (X, c) \to (Y, c')$ the arrows $h : X \to Y$ in $\mathcal{C}$, between the tips of the cones such that for every $j \in \mathcal{J}$ the triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {}_{c_j}\searrow & & \swarrow_{c'_j} \\ & Dj & \end{array} \tag{1.4}$$

The limit $(\lim D, p_j)$ of a diagram D consists of a terminal object in its category of cones. More than often one calls 'limit of D' the tip of the terminal cone, leaving the maps of the cone implicit *this is almost always harmless but slightly incorrect: the limits is composed of **both** parts.*

If a diagram D has a limit $(\lim D, p_j)$ we say that $\mathcal{C}$ *admits* the limit of D; if for a fixed $\mathcal{J}$, every $D : \mathcal{J} \to \mathcal{C}$ has a limit, we say that $\mathcal{C}$ has limits of shape $\mathcal{J}$ or that it has $\mathcal{J}$-limits; if for every element $\mathcal{J}$ of a subclass $\Phi \subseteq \mathsf{Cat}$ of categories, $\mathcal{C}$ has limits of shape $\mathcal{J}$, we say that $\mathcal{C}$ has limits of shape Φ or that it has Φ-limits. If $\mathcal{C}$ has Cat-limits, we say that $\mathcal{C}$ is (small-)complete.

Definition 1.2. In particular, a category $\mathcal{C}$ has all products if it has all limits of shape $S^\delta \to \mathcal{C}$ when S^δ is the discrete category over a set S, and $\mathcal{C}$ has equalizers if it has limits over $\mathcal{J} = \{0 \rightrightarrows 1\}$: this is because

- a diagram of shape S^δ specifies precisely a family of objects X_s in $\mathcal{C}$, one for every $s \in S$; being the terminal cone in this case means that there exists a family of arrows $p_s : \prod_s X_s \to X_s$ indexed by S with the property that
- a diagram of shape $\{0 \rightrightarrows 1\}$ specifies precisely a pair of morphisms $f, g : D_0 \to D_1$ in $\mathcal{C}$; being the terminal cone in this case means that there exists.

Theorem 1.3. The category Set of sets and functions has all products and all equalizers.

Proof.

- the product of a family of sets $\{X_s \mid s \in S\}$ is the usual Cartesian product $\prod_{s\in S} X_s$, constructed as the set of functions $S \to \bigcup_s X_s$ with the property that $f(s) \in X_s$. This allows to represent the elements of the set $\prod_{s\in S} X_s$ as *S-indexed sequences* $(x_s \mid s \in S, x_s \in X_s)$. Evidently, $\prod_{s\in S} X_s$ is equipped with projection maps $p_s : \prod_{t\in S} X_t \to X_s$ for every $s \in S$, picking the sth element of the S-sequence $(x_s \mid s \in S)$.

  The universal property of the product $\prod_{s\in S} X_s$ is spelled as follows:

  > For every set Z and family of functions $z_s : Z \to X_s$, there exists a unique $\bar{z} : Z \to \prod_{s\in S} X_s$ such that $p_s \circ \bar{z} = z_s$.

  Define $\bar{z}$ to be the function sending $\zeta \in Z$ to the S-sequence $(z_s\zeta \mid s \in S)$. Clearly this is the only possible definition so that

  $$\begin{array}{ccc} & & \prod_{s\in S} X_s \\ & \overset{\bar{z}}{\nearrow} & \downarrow{\scriptstyle p_s} \\ Z & \xrightarrow[z_s]{} & X_s \end{array} \tag{1.5}$$

  is a commutative triangle for every $s \in S$.
- the equalizer of a pair of maps $f, g : X \to Y$ consists of the subset $E = \{x \in X \mid fx = gx\} \subseteq X$; it realizes the universal property

  > For every $u : Z \to X$ such that $f \circ u = g \circ u$, there exists a unique $\bar{u} : Z \to E$ such that u equals the composition $Z \to E \hookrightarrow X$.

  Since E is just a subset of X, the universal property of $\mathsf{eq}(f, g)$ can be rephrased as follows: every $u : Z \to X$ such that $f(u(z)) = g(u(z))$ for every $z \in Z$ takes values in the subset E, defined above. This is evident, as much as it is evident that E is chosen precisely in order to satisfy this property. □

Lemma 1.4.
If $E \xrightarrow{e} A$ *is an equalizer of* $A \overset{f}{\underset{g}{\rightrightarrows}} B$ *, then the following are equivalent:*

(1) $f = g$*,*

(2) e *is an epimorphism,*

(3) e *is an isomorphism,*

(4) id_A *is an equalizer of* f *and* g*.* □

Theorem 1.5. The category Set of sets and functions has all limits.

Proof. Let $\mathcal{D}$ be a small category and $F : \mathcal{D} \to \mathsf{Set}$ a functor. For every arrow f in $\mathcal{D}$, we denote $\mathrm{s}(f)$ the source and $\mathrm{t}(f)$ the target of f; so, if $f : D \to D'$, $\mathrm{s}(f) = D$, $\mathrm{t}(f) = D'$.

We prove that the limit $\lim F$ of F is precisely the equalizer of the pair of maps

$$\prod_{D\in\mathcal{D}_0} FD \underset{\beta^F}{\overset{\alpha^F}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.6}$$

where

- $\alpha^F((x_D \mid D \in \mathcal{D})) = (x_{\mathrm{t}(f)} \mid f \in \mathcal{D}_1)$;
- $\beta^F(((x_D \mid D \in \mathcal{D})) = (Ff(x_{\mathrm{s}(f)}) \mid f \in \mathcal{D}_1)$;

This means two things: if $\lim F$ exists, then it must be the equalizer of that pair; otoh, if that pair (α, β) has an equalizer, then such is the limit of F.

We have to prove that

(1) There exists a cone

$$\lim F \xrightarrow{\bar{p}} \prod_{D\in\mathcal{D}_0} FD \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.7}$$

where e equalizes the pair α, β;

(2) such cone is terminal; this will mean two things:
   - the universal property of $\lim F$ entails the universal property of $\mathsf{eq}(\alpha, \beta)$;
   - the universal property of $\mathsf{eq}(\alpha, \beta)$ entails the universal property of $\lim F$.

   Thus, there is a unique isomorphism $\mathsf{eq}(\alpha, \beta) \cong \lim F$.

Proving 1. is easy; if $(\lim F, p_D)$ exists, all projections $p_D : \lim F \to FD$ assemble into a unique map $\bar{p} : \lim F \to \prod_D FD$ (this is the universal property of $\prod_D FD$). Note in passing that by the lemma above if $\mathcal{D}$ is a discrete category, $\prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) = \prod_{D\in\mathcal{D}_0} FD$, α, β are invertible and thus $\lim F \cong \prod_{D\in\mathcal{D}_0} FD$, as it should be.

Now, $\bar{p} : \lim F \to \prod_D FD$ equalizes (α, β), because the components p_D form a cone: the triangle of sets and functions

$$\begin{array}{ccc} & & FD \\ & \overset{p_D}{\nearrow} & \\ \lim F & & \downarrow{\scriptstyle Ff} \\ & \underset{p_{D'}}{\searrow} & \\ & & FD' \end{array} \tag{1.8}$$

is commutative, whence the fact that for all $\hat{x} \in \lim F$ and $f : D \to D'$ in $\mathcal{D}_1$ one has

$$\beta^F(\bar{p}(\hat{x})) = \big(Ff(p_{\mathrm{s}(f)}(\hat{x})) \mid f \in \mathcal{D}_1\big) = \big(p_{\mathrm{t}(f)}(\hat{x}) \mid f \in \mathcal{D}_1\big) = \alpha^F(\bar{p}(\hat{x})). \tag{1.9}$$

A similar argument for a general cone $(z : Z \to FD \mid D \in \mathcal{D}_0)$ proves that this is a cone for F if and only if it equalizes (α, β); thus, a cone for F must be a terminal cone wrt the property of equalizing (α, β); whence the conclusion. □

More generally, for a category $\mathcal{C}$ to have all limits it is necessary and sufficient that it has all (small) products and equalizers.

Theorem 1.6. The following conditions are equivalent:

- $\mathcal{C}$ has all small products and all equalizers;
- $\mathcal{C}$ is small-complete.

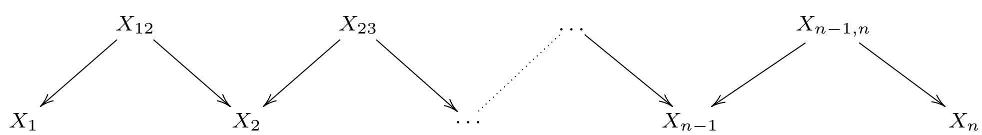
Proof. □

Exercise 1.7. The *triqualizer* $\mathsf{triq}(f,g,h)$ of functions $f,g,h : X \to Y$ is defined as the limit of the diagram

$$\mathcal{J} = \left\{ \; 0 \overset{f}{\underset{h}{\overset{g}{\rightrightarrows\!\!\to}}} 1 \; \right\} \to \mathsf{Set}. \tag{1.10}$$

- Spell out the universal property of $\mathsf{triq}(f,g,h)$;
- by virtue of Theorem 1.6 above, $\mathsf{triq}(f,g,h)$ must be expressible as an equalizer of two maps between products. How?

Exercise 1.8. An *Urizen compass* G_n is a limit of a diagram of the following form:



write the universal property for G_3 and G_4; express G_3 and G_4 as equalizers of maps between products.

2. Symmetry and associativity, interchange of limits, Yoneda and limits

The universal property of a limit entails that

2.1. Functoriality of limits.

Theorem 2.1. A natural transformation $\alpha : D \Rightarrow D'$ in the category of functors $D : \mathcal{J} \to \mathsf{Set}$, induces a morphism $\lim \alpha : \lim D \to \lim D'$ between the limits. Moreover, lim preserves identities and compositions, hence it is a functor

$$\lim : [\mathcal{J}, \mathsf{Set}] \longrightarrow \mathsf{Set} \tag{2.1}$$

The natural transformation α has components $Dj \to D'j$, and the composition $\lim D \xrightarrow{p_j} Dj \xrightarrow{\alpha_j} D'j$ is easily seen to be a cone for D'. Then, there is a unique $\bar{\alpha} : \lim D \to \lim D'$ such that each diagram

$$\begin{array}{ccc} \lim D & \xrightarrow{\bar{\alpha}} & \lim D' \\ {\scriptstyle p_j}\downarrow & & \downarrow{\scriptstyle p'_j} \\ Dj & \xrightarrow[\alpha_j]{} & D'j \end{array} \tag{2.2}$$

commutes. Uniqueness proves that $\overline{\beta \circ \alpha} = \bar{\beta} \circ \bar{\alpha}$ and $\overline{\mathrm{id}_D} = \mathrm{id}_{\lim D}$.

2.2. Yoneda and limits. One can use Yoneda lemma to express the universal property of limits; given a diagram $D : \mathcal{J} \to \mathcal{C}$, one can express the property

For all $A \in \mathcal{C}_0$, there is a (canonical, natural) isomorphism

$$\mathcal{C}(A, \lim_{j:\mathcal{J}} D_j) \cong \lim_{j:\mathcal{J}} \mathcal{C}(A, Dj) \tag{2.3}$$

(where on the right hand side we mean the limit of the composite functor

$$\begin{array}{ccccc} \mathcal{J} & \xrightarrow{D} & \mathcal{C} & \xrightarrow{\mathcal{C}(A,-)} & \mathsf{Set} \\ j & \longmapsto & Dj & \longmapsto & \mathcal{C}(A, Dj), \end{array} \tag{2.4}$$

which we know exists in the category of sets) as the representability of the functor

$$\begin{array}{rcl} \tilde{D} = \lim_{\mathcal{J}} \mathcal{C}(\_, Dj) : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathsf{Set} \\ A & \longmapsto & \lim_{\mathcal{J}} \mathcal{C}(A, Dj) \end{array} \tag{2.5}$$

Lemma 2.2. The category of elements $\mathsf{Elts}(\tilde{D})$ coincides with the category of cones for D.

Proof. We check that objects and morphisms of one category identify with objects and morphisms of the other –it is a simple matter to verify that composition and identities are the same. First, observe that $\tilde{D}A$ fits into an equalizer

$$\tilde{D}A \longrightarrow \prod_j \mathcal{C}(A, Dj) \underset{\mathcal{C}(A,Df)[\_]}{\overset{[\_]}{\rightrightarrows}} \prod_{f:x\to y} \mathcal{C}(A, Dj) \tag{2.6}$$

where $\alpha^f = [u_j : A \to Dj]^{f:x\to y} = u_y$ and $\beta^f = \mathcal{C}(A, Df)[u_j : A \to Dj] = Df \circ u_x$; but then, a family $(u_j : A \to Dj)$ equalizes α^f, β^f if and only if the triangle

$$\begin{array}{ccc} & A & \\ {}^{u_x}\swarrow & & \searrow^{u_y} \\ Dx & \xrightarrow[Df]{} & Dy \end{array} \tag{2.7}$$

commutes, and equalizes *all* α^f, β^f if and only if $(u_j : A \to Dj)$ is a cone for D with tip A; hence,

- the objects of $\mathsf{Elts}(\tilde{D})$ are pairs $(A, (u_j) \in \prod_j \mathcal{C}(A, Dj))$ such that $Df \circ u_x = u_y$ for every $f : x \to y$, i.e. families that are cones;
- a morphism $(A, (u_j)) \to (B, (w_j))$ is a morphism in the base $h : A \to B$ such that $\tilde{D}h : \tilde{D}B \to \tilde{D}A$ sends (w_j) to (u_j); but $\tilde{D}h$ acts composing each cone map $w_j B \to Dj$ with h, hence the condition reduces to $w_j \circ h = u_j$, which is exactly the condition that

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {}_{u_j}\searrow & & \swarrow_{w_j} \\ & Dj & \end{array} \tag{2.8}$$

This concludes the proof. □

Yoneda says that it is enough to know how limits are computed in Set (and in a category of functors into Set) to define them uniquely (=up to unique iso) in any category $\mathcal{C}$, via the notion of representability; the 'limit' $(\lim D, p_j)$ of a diagram

$D : \mathcal{J} \to \mathcal{C}$ consists of a terminal object in the category of elements of $\lim \mathcal{C}(\_, Dj)$ (which is *precisely* the category of cones for D), while the cone

$$p_j : \lim D \longrightarrow Dj \tag{2.9}$$

is the universal element: it corresponds to the identity of $\lim D$ in the bijection

$$\mathcal{C}(\lim D, \lim D) \cong \lim \mathcal{C}(\lim D, Dj) \tag{2.10}$$

for $A = \lim D$.

Reading (2.3) as the isomorphism

$$y(\lim D)(A) \cong \big(\lim yD\big)(A) \tag{2.11}$$

one can also argue that y (the Yoneda embedding) *preserves* all limits:

Definition 2.3. Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram; let $F : \mathcal{C} \to \mathcal{X}$ be a functor; let $\lim D$ denote the limit of D and $\lim(FD)$ the limit of the composite functor $F \circ D : \mathcal{J} \to \mathcal{X}$; then, the functoriality of F defines a cone

$$\gamma_{F,j} : F(\lim D) \longrightarrow FDj \tag{2.12}$$

whence a unique morphism, in $\mathcal{X}$, $\gamma_F : F(\lim D) \to \lim(FD)$. We say that F *preserves* $\lim D$, or that it *commutes* with $\lim D$ if γ_F is an isomorphism in $\mathcal{X}$.

Proposition 2.4. Limits in the category $[\mathcal{C}^{\text{op}}, \mathsf{Set}]$ are computed objectwise in Set: this means that given a diagram

$$D : \mathcal{J} \longrightarrow [\mathcal{C}^{\text{op}}, \mathsf{Set}] \tag{2.13}$$

its limit $\lim D$ is a functor $\mathcal{C}^{\text{op}} \to \mathsf{Set}$ defined as $(\lim_j D_j)(C) = \lim_j \big(D_j C\big)$.

Remark 2.5. The yoneda embedding

$$y : \mathcal{C} \longrightarrow [\mathcal{C}^{\text{op}}, \mathsf{Set}] \tag{2.14}$$

preserves all limits.

Evidently, F preserves all limits if and only if it preserves all products and all equalizers; it's easy to show that y preserves products and equalizers directly from the definition.

2.3. **Limits commute with limits.** The following argument will work for any category $\mathcal{C}$ be a category admitting products and equalizers, but we will spell it out just for sets; given a S-indexed family of parallel arrows

$$X_s \underset{g_s}{\overset{f_s}{\rightrightarrows}} Y_s \tag{2.15}$$

one can consider the parallel maps induced between the products using Theorem 2.1,

$$\textstyle\prod_s X_s \underset{\left(\prod_s g_s\right)}{\overset{\left(\prod_s f_s\right)}{\rightrightarrows}} \prod_s Y_s. \tag{2.16}$$

The map $\left(\prod_s f_s\right)$ is defined as the unique with the property that for every $s' \in S$, the square

$$\begin{array}{ccc} \prod_s X_s & \xrightarrow{\left(\prod_s f_s\right)} & \prod_s Y_s \\ {\scriptstyle p^X_{s'}}\downarrow & & \downarrow{\scriptstyle p^Y_{s'}} \\ X_{s'} & \xrightarrow[f_{s'}]{} & Y_{s'} \end{array} \tag{2.17}$$

commutes. Similarly, for $\left(\prod_s g_s\right)$.

Theorem 2.6. The equalizer of (2.16) above is the product $\prod_s h_s : \prod_s E_s \to \prod_s X_s$ of the equalizers $(E_s \xrightarrow{h_s} X_s \rightrightarrows Y_s \mid s \in S)$. In other words,

$$\lim_{0\rightrightarrows 1}(\lim_{S^\delta} D) \cong \lim_{S^\delta}(\lim_{0\rightrightarrows 1} D). \tag{2.18}$$

A single function $v : Z \to \prod_s X_s$ corresponds to a family of functions $(v_s : Z \to X_s \mid s \in S)$, and the fact that $\left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$ is exactly equivalent to the fact that putting $v_s = p_s \circ v$

$$\forall s \in S.(f_s \circ v_s = g_s \circ v_s) \tag{2.19}$$

All in all, every part of the following diagram does commute, for every $s \in S$:

$$\begin{array}{ccc} \prod_s E_s & \xrightarrow{p^E_s} & E_s \\ {\scriptstyle \prod_s h_s}\downarrow & & \downarrow{\scriptstyle h_s} \\ \prod_s X_s & \xrightarrow{p^X_s} & X_s \\ {\scriptstyle \left(\prod_s f_s\right)}\downarrow\downarrow{\scriptstyle \left(\prod_s g_s\right)} & & {\scriptstyle f_s}\downarrow\downarrow{\scriptstyle g_s} \\ \prod_s Y_s & \xrightarrow{p^Y_s} & Y_s \end{array} \tag{2.20}$$

Now, let's prove that in these notations

$$\left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v \iff \forall s \in S.(f_s \circ v_s = g_s \circ v_s). \tag{2.21}$$

Indeed,

- if $\left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$, then for all $s \in S$

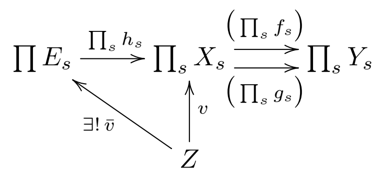
$$p^Y_s \circ \left(\prod_s f_s\right) \circ v = p^Y_s \circ \left(\prod_s g_s\right) \circ v \tag{2.22}$$

  but the LHS of this equation is $f_s \circ p^X_s \circ v$, and RHS is $g_s \circ p^X_s \circ v$.
- Conversely, if $\forall s \in S.(f_s \circ v_s = g_s \circ v_s)$, then one uses uniqueness: for all $s \in S$,

$$p^Y_s \circ \left(\prod_s f_s\right) \circ v = f_s \circ p^X_s \circ v_s = g_s \circ p^X_s \circ v_s = p^Y_s \circ \left(\prod_s g_s\right) \circ v \tag{2.23}$$

  but there exists a *unique* $w : Z \to \prod_s Y_s$ such that for all $s \in S$ one has that $p^Y_s \circ w$ equals the common value $f_s \circ p^X_s \circ v_s = g_s \circ p^X_s \circ v_s$, hence $w = \left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$.

Now, given this, each v_s equalizes (f_s, g_s), and thus factors through the equalizer $Z \xrightarrow{\bar{v}_s} E_s \xrightarrow{h_s} X_s$ so that $v_s = h_s \circ \bar{v}_s$; by the universal property of $\prod_s E_s$, there is a unique map $\bar{v} : Z \to \prod_s E_s$, which must be so that $\left(\prod_s h_s\right) \circ \bar{v} = v$:



(2.24)

2.4. **Unitality and associativity of products.**

One can use the universal property of products to exhibit natural isomorphisms

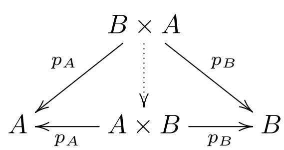
$$A \times 1 \cong A \cong 1 \times A \qquad (A \times B) \times C \cong A \times (B \times C) \qquad A \times B \cong B \times A \tag{2.25}$$

Indeed, we just have to show that A has the same universal property of $A \times 1$, that $B \times A$ has the same universal property of $A \times B$, etc.

- A comes equipped with two projections

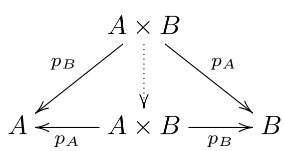
$$1 \longleftarrow A = A \tag{2.26}$$

  which satisfy the universal property $1 \times A$ is required to have.
- Consider the universal problem



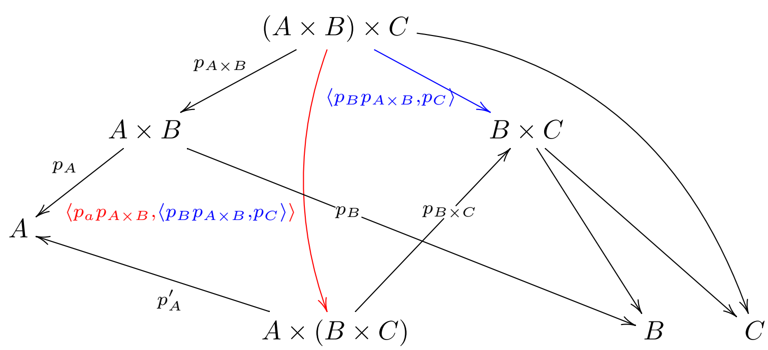
(2.27)

  It must have a unique solution $\sigma = \langle p_B, p_A \rangle$. Uniqueness of $\mathrm{id}_{A \times B}$ solving the universal problem



(2.28)

  implies that $\sigma \circ \sigma = \mathrm{id}_{A \times B}$.
- Associativity uses diagram chasing at its full potential:



(2.29)

  the blue arrow $\langle p_B p_{A\times B}, p_C \rangle$ is induced by its arguments $p_B p_{A\times B}, p_C$; similarly for the red arrow $\langle p_a p_{A\times B}, \langle p_B p_{A\times B}, p_C \rangle\rangle$. In a similar fashion one can induce an arrow $A \times (B \times C) \to (A \times B) \times C$; these two arrows are mutually inverse to each other, and unique.

Exercise 2.7. Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram; show that if $\mathcal{J}$ has an initial object 0, then $\lim D = D0$.