Homotopical algebra of C^*-algebras
January 16, 2013
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Bundles by A. T. FOMENKO

Abstract

The plan of this note is to present (in a way that is particularly self-contained to those who know little Category Theory) the ideas in **[Uuye]**'s paper, giving a precise account of the methods in it, and using the whole and well-established machinery of *Homotopical Algebra* to give C^*-**Alg** an homotopical calculus.

Contents

> Perhaps it is not frivolous to say that [...] model categories are "convenient categories to do homotopical algebra in", and to view them as *non-abelian counterparts of Grothendieck abelian categories*.
>
> Tibor Beke

INTRODUCTION. A *model category* is a category **C** endowed with three suitably interacting classes of morphisms, *weak equivalences*, *fibrations* and *cofibrations*, letting us study Homotopy Theory in a purely arrow-theoretic setting.

The definition of model categories as an abstract setting to do Homotopy Theory is due to **[Quillen]**'s seminal work (even if a tentative of "abstracting Homotopy Theory" dates back to Kan's series of articles on Simplicial Homotopy published since 1956 by the *Proceedings* of the National Academy of Sciences of the USA), and the philosophy behind that definition is

> Thou shalt astray a minimal set of properties that permit to extend Homotopy Theory to categories other than topological spaces; moreover, thou shalt try to *internalize* classical homotopy-theoretical notions (the theory of fundamental groups and higher homotopy groups, stable homotopy, action of the π_1 on the fibers of a space, the behaviour of a covering map with respect to paths and homotopies, ...) in a suitable "category with weak equivalences".

Roughly speaking, a weak equivalence in a category **C** is an arrow in a certain sense "as similar as possible" to an isomorphism (in classical Homological Algebra there exists a well-established notion of *quasi-isomorphism*), and what we want to do is to pass in a setting (the *homotopy category* of **C**, $\mathrm{Ho}(\mathbf{C})$) where this arrow is a real isomorphism, adding the inverse it lacks: this apparatus willingly resembles the notion of (weak) homotopy equivalence in Algebraic Topology, where such maps are continuous functions $f\colon X \to Y$ inducing isomorphisms between all homotopy groups. The purely formal procedure of inversion of all quasi-isomorphisms falls under the name of *localization theory*, and it has been introduced by **[Zisman]** in their famous book: weak equivalences are all we need, or in a few words

all that matters is what we want to invert,

in the sense that *any* category with a distinguished class of weak equivalences can be endowed with an "homotopical calculus" which allows us to define *homotopy invariants of objects*. The whole machinery gravitating around weak equivalences serves to avoid certain annoying pathologies: fibrations and cofibrations work in sinergy ensuring that the localized category $\mathrm{Ho}(\mathbf{C}) =: \mathbf{C}[\mathrm{WK}^{-1}]$ is not as badly-behaved as it might happen (set-theoretic issues can prevent $\mathrm{Ob}(\mathrm{Ho}(\mathbf{C}))$ from being a set). They also ensure that we can figure the -highly untractable-set $\hom_{\mathrm{Ho}(\mathbf{C})}(A,Y)$ of arrows between A and Y in the localized category to be the set (and even before, to be *a* set) of (abstract) *homotopy classes* of arrows between A and Y (cfr. **[Brown]** as cited in **[Uuye]**, Corollary **1.19**).

It is a truism (or perhaps the proof that our machinery is really working) that the archetypal example of a model category is **Top** (let's call with this name a monoidal closed subcategory of spaces, *suitable for Homotopy Theory* in the sense of **[Steenrod]**), for example *compactly generated weakly Hausdorff spaces* (cfr. **[May]**): what really matters is a *monoidal closed* structure). In fact there are various[1] homotopical structures on **Top**, all of which recognize a weak equivalence $f\colon X \to Y$ as a map inducing isomorphisms on (all, infinitely many, some) homotopy groups, $\pi_n(X) \cong \pi_n(Y)$; what changes from one structure to another is what we call a fibration and a cofibration, in order to mantain mutual lifting properties and stability conditions (see Definition **5.3**).

These ideas showed to be extremely fruitful in studying categories of "things that resemble spaces" and *structured spaces*, keeping track of their structure in the step-by-step construction of the desired homotopy invariants; so in a certain

[1] http://ncatlab.org/nlab/show/homotopy+n-type

sense it is natural to apply this complicated machinery to the category C^***-Alg**: what's nearer to a space –albeit not being one– than a commutative unital C^*-algebra? Recall that Gel'fand-Naimark's theorem asserts an (anti-)equivalence of categories

$$\mathrm{C}^*\textbf{-Alg}_c \cong \textbf{LCHaus}$$

(it is worth to stress the importance of this equivalence in translating many "difficult" problems on one side into "simple(r)" problems on the other: see Theorem **5.2** in the last Section of this exposition).

Starting from this we shouldn't be surprised by the existence of homotopical methods in C^*-algebra theory. Hence it should be natural to spend a considerable effort to endow C^***-Alg** with a model structure, maybe exploiting one of the various pre-existing model structures on **Top**: this is what **[Uuye]** proposed in his article.

The main problem is that the category of C^*-algebras admits an homotopical calculus which can't be extended to a full model structure in the sense of **[Quillen]**. This is precisely Theorem **5.2**, which we take from **[Uuye]** repeating an unpublished argument by Andersen and Grodal; the plan to overcome this difficulty is to seek for a weaker form of Homotopical Calculus, still fitting our needs. To this end, the main reference is **[Brown]**'s thesis, which laid the foundations of this weaker abstract Homotopy Theory, based on the notion of *category with fibrant objects*. Instead of looking for a full model structure on C^***-Alg** we seek for a *fibrant* one, exploiting the track drawn by **[Uuye]**'s paper, which is the main reference of the talk together with Brown's thesis.

Once noticed that we can find a fibrant structure on C^***-Alg**, but that it doesn't come from a Quillen model structure, the obvious question that may arise is

Does the category C^***-Alg** admits a suitable, different model structure?

A tentative answer can be found in **[Østvær]**'s paper, where the category C^***-Alg** is embedded in $\textbf{Sets}^{\mathrm{C}^*\textbf{-Alg}}$ via the (co-)Yoneda functor (obtaining a category of C^*-*spaces*), and then enriching this copresheaf category over the category of *cubical sets*[2], obtaining *cubical* C^*-*spaces*, denoted $\mathrm{C}^{\square}$-**Spc**; this category admits a Quillen model structure, and **[Østvær]** studies its homotopy category in the stable and unstable version, mainly using the methods introduced in section **3**.

Another possibility is to *categorify* the notion of C^*-algebra introducing the (2-)category of C^*-*categories*, which are –roughly speaking– categories enriched over the symmetric monoidal category C^***-Alg**; we refer the interested reader to the final chapter(s) of **[Warner]**'s monography (which is particularly well-written from the point of view of Category Theory): it is interesting to notice that this model structure on C^*-**Cat** is intimately linked to the "canonical" one on the categories **Cat**/**Gpd** of small categories/groupoids (and functors between them; weak equivalences are categorical equivalences): the interested reader can

[2] http://ncatlab.org/nlab/show/cubical+set

again refer [**Warner**]'s monography and the paper [**Dell'Ambrogio,2**] (the most updated version of it is 20 days old at the moment we are writing this note).

The ideal reader of this exposition should have a little acquaintance with classical Homotopy Theory, even if we will try to keep at minimum level the prerequisites needed (action of the π_1 on the fibers of a covering, the related Galois' theory, a little confidence with Spanier's "functorial topology"...). This reading is strongly advised to those cherishing for *abstract nonsense* arguments, because of its evident categorical flavour.

1 Categories with Fibrant Objects.

> Comme il s'agit de catégories il y a des flèches, des *diagrammes*. Il y a peu, très peu, ou pas du tout de calculs.
>
> J. Roubaud

The goal of this preliminary section is to establish suitable properties of a "category in which to develop Homotopy Theory". The typical category $\mathbf{C}$ we will consider admits at least any finite limit and colimit: notice that this entails $\mathbf{C}$ admits a initial and terminal object.

Definition 1.1 (Category with Weak Equivalences)**.** A *category with weak equivalences* is a category $\mathbf{C}$ with a distinguished class of morphisms $\mathrm{WK} \subseteq \mathrm{Mor}(\mathbf{C})$ which contains all isomorphisms of $\mathbf{C}$, which is closed under composition and which satisfies the *two-out-of-three property*:

> For f, g any two composable morphisms of $\mathbf{C}$, if any two of $\{f, g, g \circ f\}$ are in WK, then so is the third.

Definition 1.2 (Fibrations and Path Objects)**.** Let $(\mathbf{C}, \mathrm{WK})$ be a category with weak equivalences, and consider another class of morphisms $\mathrm{FIB} \subset \mathrm{Mor}(\mathbf{C})$, to be called *fibrations*. A morphism $f \in \mathrm{WK} \cap \mathrm{FIB}$ will be called an *acyclic (or trivial, or aspherical) fibration*.

A *path object* for an object $B \in \mathrm{Ob}_{\mathbf{C}}$ consists of a triple

$$(B^I, s, \langle d_0, d_1 \rangle) \in \mathrm{Ob}_{\mathbf{C}} \times \hom(B, B^I) \times \hom(B^I, B \times B),$$

where the composition

$$B \xrightarrow{\quad s \quad} B^I \xrightarrow{\langle d_0, d_1 \rangle} B \times B$$

is a factorization of the canonical arrow $\Delta \colon B \to B \times B$ obtained by the universal property of the product in the diagram

$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{1} & \vdots{\scriptstyle\Delta} & {}^{1}\searrow & \\ B & \xleftarrow[\pi_2]{} & B \times B & \xrightarrow[\pi_1]{} & B. \end{array}$$

and where the arrow s is a weak equivalence, and the arrow $\langle d_0, d_1 \rangle$ is a fibration.

Notation. We will often write $\xrightarrow{\sim}$ to denote a weak equivalence, and $\twoheadrightarrow$ to denote a fibration.

Definition 1.3 (Category with Fibrant Objects)**.** A *category with fibrant objects* (cfo for short) is a triple $(\mathbf{C}, \text{WK}, \text{FIB})$ where $(\mathbf{C}, \text{WK})$ is a category with weak equivalences, and where FIB is a class of maps such that

CF1 (closure) Any isomorphism is a fibration; FIB is closed under composition.

CF2 (base change stability) FIB and FIB $\cap$ WK are *stable under pullback*: if $f\colon A \twoheadrightarrow B$ is a(n acyclic) fibration, and $u\colon X \to B$ is any arrow, then in the diagram

$$\begin{array}{ccc} A \times_B X & \longrightarrow & A \\ \big\downarrow & & \big\downarrow f \\ X & \xrightarrow[u]{} & B \end{array}$$

the arrow $A \times_B X \to X$ is again a(n acyclic) fibration.

CF3 (Existence of "enough" path objects) For all $B \in \text{Ob}_{\mathbf{C}}$ there exists at least a path object $(B^{\boldsymbol{I}}, s, \langle d_0, d_1 \rangle)$ (possibly non-functorial in B).

CF4 ("Fibrance") Every object is *fibrant*, that is the unique arrow $B \to *$ to the terminal object is a fibration.

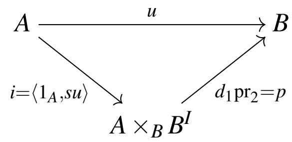
REMARK 1 : *In a cfo the projection maps $A \times B \to A, B$ are fibrations –they can be obtained via a pullback*

$$\begin{array}{ccc} A \times B & \longrightarrow & A \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & *. \end{array}$$

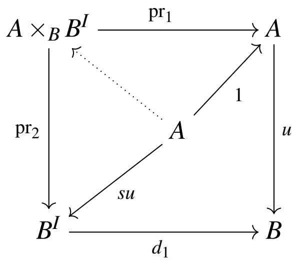
Hence in a cfo, for any B the composition maps $d_i = \pi_i \circ \langle d_0, d_1 \rangle$, $i = 0, 1$, are acyclic fibrations in $\hom(B^{\boldsymbol{I}}, B)$.

Lemma 1.1 (Brown's Factorization Lemma)**.** The couple $(\text{WK}_{\mathbf{C}}, \text{FIB}_{\mathbf{C}})$ is a *factorization system* in $\mathbf{C}$, i.e. any arrow $u \in \text{Mor}(\mathbf{C})$ can be factored as the composition $p \circ i$ of a weak equivalence i and a fibration p; this fibration is acyclic if and only if u was a weak equivalence.

Proof. Choose once and for all a path object $(B^I, s, \langle d_0, d_1\rangle)$ for $B = \operatorname{cod} u$. It is always possible to factor u as a composition



where $A \to A \times_B B^I$ can be obtained from the diagram



using the universal property of pullback. The composition

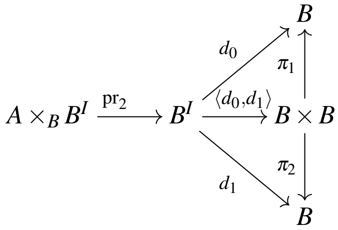
$$p \circ i = d_1 \circ s \circ u = p_1 \circ \langle d_1, d_2\rangle \circ s \circ u = 1_B \circ u = u,$$

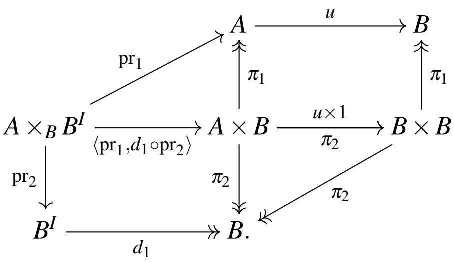
hence (i, p) really factors u; the composition

$$\mathrm{pr}_1 \circ i = p_1 \circ (1_A, s \circ u) = 1_A,$$

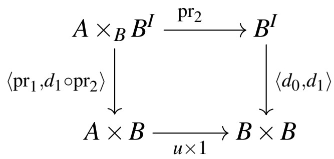
shows that i is right inverse to an acyclic fibration, hence by 2-out-of-3 it is a weak equivalence.

Now we have to show that p is a fibration. Consider the two diagrams





Let's show that the square



is commutative and cartesian (i.e. a pullback square); this entails $(\mathrm{pr}_1, d_1 \circ \mathrm{pr}_2)$ is a fibration, hence $\pi_1 \circ (\mathrm{pr}_1, d_1 \circ \mathrm{pr}_2) = d_1 \circ \mathrm{pr}_2 = p$ is a fibration; to this end let's notice that

$$\begin{aligned} \pi_1 \circ \langle d_0, d_1 \rangle \circ \mathrm{pr}_2 &= d_0 \circ \mathrm{pr}_2 = u \circ \mathrm{pr}_1 \\ \pi_1 \circ (u \times 1) \circ \langle \mathrm{pr}_1, d_1 \circ \mathrm{pr}_2 \rangle &= u \circ \pi_1 \circ \langle \mathrm{pr}_1, d_1 \circ \mathrm{pr}_2 \rangle = u \circ \mathrm{pr}_1 \\ \pi_2 \circ \langle d_0, d_1 \rangle \circ \mathrm{pr}_2 &= d_1 \circ \mathrm{pr}_2 = \pi_2 \circ \langle \mathrm{pr}_1, d_1 \circ \mathrm{pr}_2 \rangle. \end{aligned}$$

□

2 THE HOMOTOPY CATEGORY.

> The fundamental problem of Algebraic Topology can be stated in Danish: 'Er en smultring en berlinerbolle?'
>
> A. Stacey

Definition 2.1 (Homotopy Category)**.** Let $(\mathbf{C}, \mathrm{WK})$ be a category with weak equivalences; we call *homotopy category* of $\mathbf{C}$ the category $\mathrm{Ho}(\mathbf{C}) = \mathbf{C}[\mathrm{WK}^{-1}]$ obtained localizing $\mathbf{C}$ over the multiplicative system of weak equivalences (see **[Zisman]** and more generally **[Krause]**'s review for a precise account about localization theory).

Let $\mathbf{C}$ be a cfo. Then the *fundamental theorem of homotopical algebra* asserts that the homotopy category $\mathrm{Ho}(\mathbf{C})$ can be explicitly described via *homotopies* between maps.

Definition 2.2 (Homotopy relation)**.** Two arrows $f, g \colon A \rightrightarrows B$ are called *(right) homotopic* if there exists a path object B^I and a third arrow $h \colon A \to B^I$ such that $d_0 \circ h = f, d_1 \circ h = g$. We write $f \rtimes_h g$ to denote that f, g are right homotopic via h.

Being right homotopic is an equivalence relation on $\hom(A, B)$; reflexivity can be obtained choosing $h = s \circ f \colon A \to B^I$ for a fixed path object $(B^I, s, \langle d_0, d_1 \rangle)$. Simmetry holds because of the presence of the arrow $\sigma_B \colon B \times B \to B \times B$ such that $\sigma \circ pi_1 = \pi_2, \sigma \circ \pi_2 = \pi_1$: if $f \rtimes_h g$, then $g \rtimes_{\sigma h} f$. Transitivity requires a more involved argument.

REMARK 2 : *Homotopic maps become equal in* $\mathrm{Ho}(\boldsymbol{C})$, *because if we denote* $\gamma \colon \boldsymbol{C} \to \mathrm{Ho}(\boldsymbol{C})$ *the localization functor then*

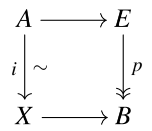
$$\gamma(f) = \gamma(d_0)\gamma(h) = \gamma(s)^{-1}\gamma(h) = \gamma(d_1)\gamma(h) = \gamma(g).$$

It is straightforward that if $f \rtimes_h g$, then $f \circ u \rtimes_{h \circ u} g \circ u$ for any $u \colon C \to A$ composable with f, g; on the other hand, there can be no homotopy k between $v \circ f$ and $v \circ g$. But there is still hope:

Proposition 2.1. Suppose $v\colon B \to C$, and $f \rtimes_h g$, then there are an acyclic fibration $t\colon A' \to A$ and a homotopy $k\colon A' \to C^I$ such that $uft \rtimes_k ugt$.

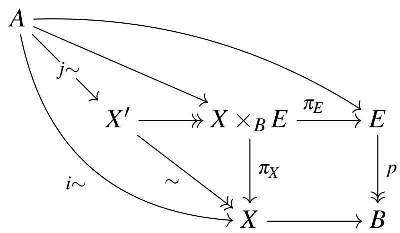
We need to collect some preliminaries to prove this proposition:

Lemma 2.1. Suppose that in the square



$(i,p) \in \mathrm{WK} \times \mathrm{FIB}$. Then i can be factored as a composition $A \to X' \overset{\sim}{\twoheadrightarrow} X$.

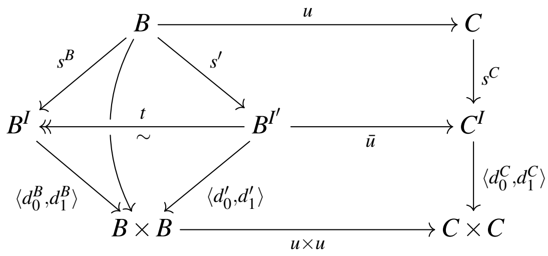
Proof. Apply Brown's Factorization Lemma to the (unique by universal property) arrow $u\colon A \to A \times_B E$ we obtain



Now, π_X is a fibration (it is obtained by pulling back via p), and $\pi_X \circ r$ is the acyclic fibration we want: $\pi_X \circ r \circ j = i$ and j are weak equivalences, and the 2-out-of-3 property concludes the argument. $\square$

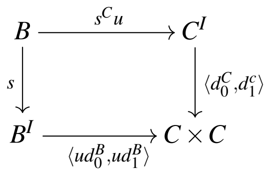
The following Lemma is not only necessary to show what we stated before, but also to ensure the existence and functoriality of the loop object in Theorem **2.3**.

Lemma 2.2. Let $u\colon B \to C$ be an arrow in a cfo $\mathbf{C}$ and choose path objects $(B^I, s^B, \langle d_0^B, d_1^B\rangle)$, $(C^I, s^C, \langle d_0^C, d_1^C\rangle)$. Then there exist a second path object $(B^{I'}, s', \langle d_0', d_1'\rangle)$ for B, an acyclic fibration $t\colon B^{I'} \to B^I$ and an arrow $\bar{u}\colon B^{I'} \to C^I$ such that the diagram

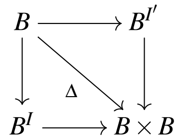


commutes.

Proof. Apply the previous Lemma to the square

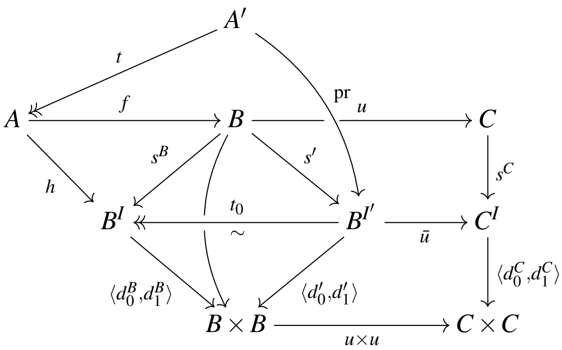


and notice that this gives a factorization $B \to Z \to B^I$; let $Z = B^{I'}$, and notice that it is a path object because the square



commutes (Δ is the curved vertical arrow). □

Proof of Prop. 2.1. Let $f \rtimes_h g$ via $h\colon A \to B^I$, and consider the diagram



where $A' \cong A \times_{B^I} B^{I'}$, $B^{I'}$ being obtained as in the second Lemma. Then the composition $A' \xrightarrow{\text{pr}} B^{I'} \xrightarrow{\bar{u}} C^I$ acts like an homotpy between uft e ugt (simply follow suitable arrows in the previous intricated diagram). □

Now, the homotopy category $\mathrm{Ho}(\mathbf{C})$ can be in a certain sense *approximated* with a category $\pi\mathbf{C}$ obtained from $\mathbf{C}$ in the following way:

- Objects in $\pi\mathbf{C}$ are the same as in $\mathbf{C}$ and $\mathrm{Ho}(\mathbf{C})$;
- Morphisms between A and B in $\pi\mathbf{C}$ are collected in the set of homotopy classes of arrows $f, g\colon A \rightrightarrows B$:

  $$\hom_{\pi\mathbf{C}}(A,B) = \hom_{\mathbf{C}}(A,B)/\rtimes$$

  where $f \rtimes g$ if and only if there exist a weak equivalence $t\colon X \to A$ and a homotopy h such that $f \circ t \rtimes_h g \circ t$.

$\mathrm{Ho}(\mathbf{C})$ is now obtained by localizing $\pi\mathbf{C}$ with respect to the class $\mathrm{W}\textsc{k}_{\pi\mathbf{C}}$ of (homotopy classes of) weak equivalences, and this localization can be made explicit by the fact that

Proposition 2.2. The class $\mathrm{W}\textsc{k}_{\pi\mathbf{C}}$ admits a right calculus of fractions in the sense of **[Zisman]**, namely

- Any diagram $A \to C \xleftarrow[\sim]{t} B$ admits a completion to a commutative square (in $\pi\mathbf{C}$)

$$\begin{array}{ccc} A' & \dashrightarrow & B \\ {\scriptstyle t'}\Big\downarrow{\scriptstyle \sim} & & \Big\downarrow{\scriptstyle t} \\ A & \longrightarrow & C \end{array}$$

- Given two arrows $f,g\colon A \rightrightarrows B$, there exists a weak equivalence equalizing themm if and only if there exists a weak equivalence coequalizing them.

Now we can deduce that the relation $\rtimes_{(-)}$ is a real congruence in $\hom_{\mathbf{C}}(A,B)$. The proof exhibiting a calculus of fraction for $\mathrm{W}\textsc{k}_{\pi\mathbf{C}}$ is based on the following

Lemma 2.3. Given any diagram $A \xrightarrow{u} C \xleftarrow{v} B$ the projection $A \times_C C^I \times_C B \to A$ is a fibration, acyclic if $v \in \mathrm{W}\textsc{k}$.

which we accept without proof (see **[Brown]**, Proposition 2).

At this point you probably have had enough of this endless plethora of abstract-nonsense arguments, but thanks to it we are able to state the following corollary.

Let $A, B \in \mathrm{Ob}_{\mathbf{C}}$, and

$$[A,B]_{\mathbf{C}} := \varinjlim_{\pi\mathbf{C}/A} \hom_{\pi\mathbf{C}}(-,B)$$

where $\pi\mathbf{C}/A$ contains as objects $[t]\colon X \to A$ homotopy classes of weak equivalences in $\pi\mathbf{C}$, and an arrow $[t\colon X \to A] \to [s\colon Y \to A]$ consists of a homotopy class of arrows $X \to Y$ making the obvious triangle commute.

Theorem 2.1. Let $\mathbf{C}$ be a cfo. Then, for any $A, B \in \mathrm{Ob}_{\mathbf{C}}$ there exists a canonical isomorphism

$$\hom_{\mathrm{Ho}(\mathbf{C})}(A,B) \cong [A,B]_{\mathbf{C}}.$$

In particular if $\gamma\colon \mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ is the localization functor, then

- Any arrow $f\colon A \to B$ in $\mathrm{Ho}(\mathbf{C})$ can be written as a *right fraction* $\gamma(f') \circ \gamma(t)^{-1}$ where $t \in \mathrm{W}\textsc{k}$:

$$\begin{array}{ccccc} & & A' & & \\ & {\scriptstyle t}\swarrow & & \searrow{\scriptstyle f'} & \\ A & & \underset{f}{\dashrightarrow} & & B \end{array}$$

- If $f,g\colon A \rightrightarrows B$, then $\gamma(f)=\gamma(g)$ if and only if there exists a weak equivalence t which coequalizes both.

REMARK 3 : *It's easy to show, by directly checking every axiom, that if* $\mathbf{C}$ *is a cfo, then the subcategory* $\mathbf{C}_B$ *of the slice category* $\mathbf{C}/B$ *obtained by taking fibrations* $X \twoheadrightarrow B$ *as objects is again a cfo (the only non-immediate axiom to verify is the existence of enough path objects: use the Factorization Lemma).*

Lemma 2.4. Let $\mathbf{C}$ be a cfo. Then for any $u\colon B' \to B$ the functor $u^*\colon \mathbf{C}_B \to C_{B'}$ preserves fibrations and weak equivalences.

Proof. Factorization Lemma applied to $\mathbf{C}_B$ ensures that without loss of generality it suffices to show that u^* preserves (acyclic) fibrations (any weak equivalence can be written as the composition of an acyclic fibration and a weak equivalence). But this follows entirely from functoriality of u^*, sending $X \to B$ in its pullback via u:

$$\begin{array}{ccc} u^*(X\to B) & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \xrightarrow[u]{} & B. \end{array}$$

It is straightforward that any pullback is completely determined by the cone $\bullet \to \bullet \leftarrow \bullet$ defining it, hence if $p\colon (E_1 \overset{e_1}{\twoheadrightarrow} B) \twoheadrightarrow (E_2 \overset{e_2}{\twoheadrightarrow} B)$ is a fibration in $\mathbf{C}_B$, juxtaposing the two squares

$$\begin{array}{ccc} U & \longrightarrow & E_1 \\ \downarrow & & \downarrow{\scriptstyle p} \\ B'\times_B E_2 & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ B' & \xrightarrow[u]{} & B \end{array} \qquad (U \to B',\ E_1 \xrightarrow{e_1} B)$$

one has $U = \big(B'\times_B E_2\big)\times_{E_2} E_1 \cong B'\times_B E_1$. Hence by axiom [CF2] we can deduce $U \twoheadrightarrow B'$, and this arrow is an acyclic cofibration if p was. $\square$

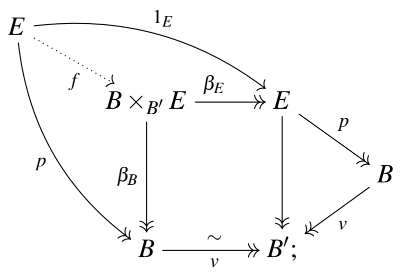
Lemma 2.5. Change of base of a weak equivalence with respect to a fibration is a weak equivalence.

Proof. Let $u\colon B' \to B$ be the weak equivalence and $p\colon E \to b$ the fibration; we have to show that in the cartesian square

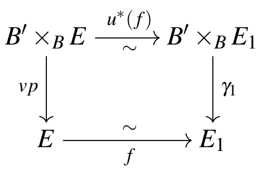
$$\begin{array}{ccc} B'\times_B E & \xrightarrow{\alpha_{B'}} & B' \\ {\scriptstyle \alpha_E}\downarrow & & \downarrow{\scriptstyle \sim\, u} \\ E & \xrightarrow[p]{} & B \end{array}$$

the arrow α_E is a weak equivalence.

Now, without loss of generality the arrow u can be considered the inverse of an acyclic fibration $v\colon B' \to B$ (Factorization Lemma). Hence we can consider the pullback of v and $v \circ p$:



$f = (p, 1_E)$ is the unique arrow factoring the two morphisms to the factors of the fibered product. It is a weak equivalence by 2-out-of-3 ($\beta_E \circ f = 1_E$). Now consider the diagram

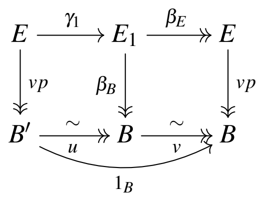


where $\gamma_1 \colon E \cong B' \times_B E_1 \to E_1$ is obtained by pulling back u via β_B.

Thanks to the previous Lemma $u^*(f)$ is a weak equivalence, hence showing that also γ_1 is suffices to conclude (one can invoke again 2-out-of-3 property). On the other hand this follows from

$$B' \times_B E_1 \cong B' \times_B \left(B \times_{B'} E\right) \cong E$$

i.e. from the fact that in the commutative diagram



one has $\beta_E \circ \gamma_1 = 1_E$, and $\beta_E \in \text{WK} \cap \text{FIB}$. $\square$

We call a cfo **C** pointed if **C** admits a zero object. The following two highly technical (and boring) results are necessary to show that in any pointed cfo **C** we can build the *loop object* of an object B, exploiting the existence of enough path objects in **C**: the construction is a (hopefully) straightforward abstraction of the existence of the loop space ΩX in **Top** for any space X.

Lemma 2.6 (Homotopical 5 lemma)**.** Let **C** be a pointed cfo, and

$$\begin{array}{ccccc} F' & \xrightarrow{i'} & E' & \xrightarrow{p'} & B' \\ {\scriptstyle h}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow[i]{} & E & \xrightarrow[p]{} & B \end{array}$$

a commutative diagram where $p, p' \in \mathrm{FIB}$, and i, i' are the *inclusions of the typical fibre*, i.e. the natural morphisms arising from the pullback

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & B. \end{array}$$

If $f, g \in \mathrm{WK}$, then also h is.

Proof. Omitted (see **[Brown]**, Lemma **3**, pag. **429**). ☐

Lemma 2.7. Let **C** be a pointed cfo, $p_1 \colon E_1 \twoheadrightarrow B$, $p_2 \colon E_2 \twoheadrightarrow C \in \mathrm{FIB}$ with typical fibers F_1, F_2, $u \colon B \to C$ any arrow and $f, g \colon E_1 \rightrightarrows E_2$ such that $p_2 \circ f = p_2 \circ g = u \circ p_1$, as in the following diagram:

$$\begin{array}{ccccc} E_1' & \xrightarrow[\sim]{t} & E_1 & \overset{f}{\underset{g}{\rightrightarrows}} & E_2 \\ & & {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ & & B & \xrightarrow[u]{} & C. \end{array}$$

If $t \in \mathrm{WK}$ equalizes both f and g then the image of f and the image of g via the localization functor coincide in $\mathrm{Ho}(\mathbf{C})$.

Proof. Omitted (see **[Brown]**, Lemma **4**, pag. **429**). ☐

2.1 The loop-object Functor.

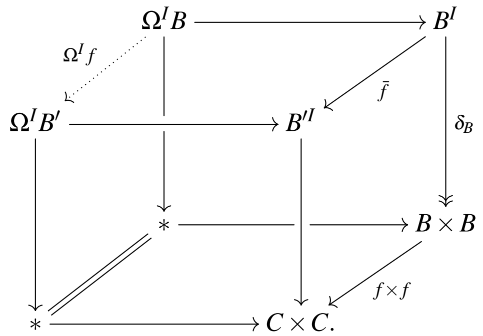
Definition 2.3. Let **C** be a pointed cfo. Define the *loop object* of $B \in \mathrm{Ob}_{\mathbf{C}}$, denoted $\Omega^I(B)$, as the typical fibre of $\langle d_0, d_1 \rangle \colon B^I \to B \times B$, once a particular path object B^I has been chosen for B.

Theorem 2.2. The correspondence $B \mapsto \Omega^I(B)$ defines a functor $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$. The loop object of B is an internal group in $\mathrm{Ho}(\mathbf{C})$; iterated loop objects $\Omega^k(B)$ are internal *abelian* groups.

Proof. Part of the proof is obviously devoted to show that the definition of $\Omega^I B$ is well posed (i.e., independent from the choice of B^I): we want to show that the correspondence $B \mapsto \Omega^I(B)$ defines a functor $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$, and for any two path objects $B^I, B^{I'}$ of the same object B, there exists a weak equivalence $\Omega^I(B) \xrightarrow{\sim} \Omega^{I'}(B)$.

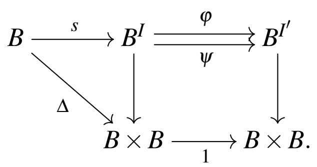
Once you finished reading this sketch of proof, notice how many times we exploited Lemma **2.2**.

Denote $(B^I, s, \langle d_0, d_1\rangle)$ the chosen path object for B, and define $\Omega^I B$ to be the typical fiber of $B^I \to B \times B$. Given any arrow $f\colon B \to B'$, the universal property of pullbacks (in particular the functoriality of the construction) entails that there exists an arrow $\hat{f}\colon \Omega^I B \to \Omega^I B'$ in the following diagram:



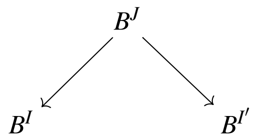
This arrow induces the dotted one by functoriality. Let's show that $\Omega^I(B)$ doesn't depend on the choice of B^I.

Suppose there exists an arrow $B^I \to B^{I'}$ between two *different* path objects for B; hence Lemma 2.2 implies that there exists an induced arrow $B^I \to B^{I'}$, which is a weak equivalence by Lemma 2.4, becoming an isomorphism in $\mathrm{Ho}(\mathbf{C})$: $\varphi\colon \Omega^I B \cong \Omega^{I'} B$ in $\mathrm{Ho}(\mathbf{C})$. Notice that this equivalence is unique using Lemma 2.7 over the diagram



Hence $\Omega^I B \cong \Omega^{I'} B$ *in a canonical fashion*.

Suppose now that the arrow $B^I \to B^{I'}$ doesn't exist, and create it by *standard cofiltration*: Lemma 2.2 implies the existence of a roof of the form

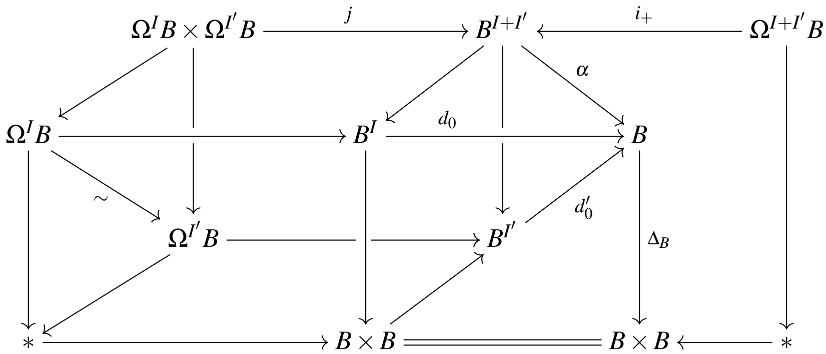


and ensures that arrows can be chosen to be weak equivalences (this is not so astonishing: any two path objects of the same object have to become isomorphic in the homotopy category, turning the correspondence $B \mapsto B^I$ into a functor. Having a right calculus of fractions allows exactly to represent an isomorphism $B^I \to B^{I'}$ as a suitable roof). We now reduced to the previous case, and we can deduce the existence of two canonical isomorphisms

$$\Omega^I B \overset{\text{can}}{\cong} \Omega^J B \overset{\text{can}}{\cong} \Omega^{I'} B =: \Omega B.$$

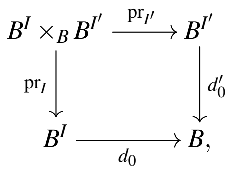
whose composition is the desired identification. Let's now show that $\Omega^I B = \Omega B$ is a internal group in the homotopy category.

Consider two path objects $B^I, B^{I'}$ for B, and refer your notations to the following diagram:

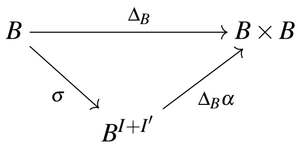


Every face in the cube is a pullback (vertices are uniquely determined by the directions of arrows), and also the right rectangle is a pullback with vertex $\Omega^{I+I'} B$.

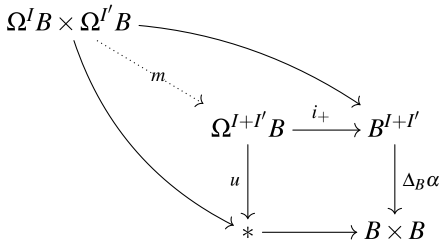
The object $B^{I+I'} = B^I \times_B B^{I'}$ can be defined via



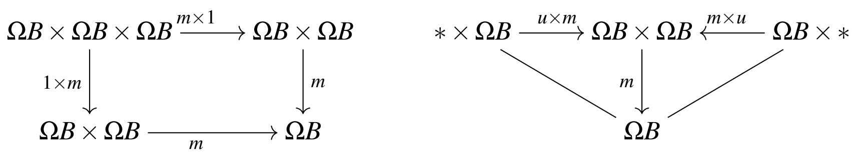
and it can be showed that it is a third path object for B: consider the diagram



(σ is obtained by universal property of the pullback applied to the pair (s, s')); the composition $\Delta_B \circ \alpha \circ \sigma$ equals $\Delta_B \circ d_0 \circ \mathrm{pr}_I \circ \sigma = \Delta_B \circ d_0 \circ s = \Delta_B$. A simple diagram chase now shows the existence of a unique map $\Omega^I \times \Omega^{I'} B \to \Omega^{I+I'} B$ completing the dotted arrow, i.e. a unique $m\colon \Omega B \times \Omega B \to \Omega B$.



Similar methods can be used to show that the diagrams



commute in the homotopy category ($u\colon * \to \Omega B$ is the unique possible arrow). Finally, the commutativity of $\Omega(\Omega(B))$ follows from Eckmann-Hilton argument, once noticed that... □

3 STABLE HOMOTOPY.

The following procedure is a general construction with which we can extend a category $\mathbf{E}$ to a bigger one $\mathcal{SW}_\Omega(\mathbf{E})$ where a given endofunctor Ω is invertible. The category $\mathcal{SW}_\Omega(\mathbf{E})$ is called the *Spanier-Whitehead category* of $\mathbf{E}$. Refer to **[Dell'Ambrogio]**'s *Diplomarbeit* for more information about the general theory of Spanier-Whitehead construction.

From now on we will consider the case of a pointed cfo $\mathbf{C}$ and $\mathbf{E} = \mathrm{Ho}(\mathbf{C})$, Ω the loop-object functor. The category we are going to exhibit is $\mathcal{SW}_\Omega(\mathrm{Ho}(\mathbf{C})) = \mathrm{SHo}(\mathbf{C})$.

Definition 3.1 (Stable Homotopy Category of a cfo)**.** Let's call the category we want to build the *stable homotopy category* $\mathrm{SHo}(\mathbf{C})$, pictorially denoted $\mathrm{Ho}(\mathbf{C})[\Omega^{-1}]$. Its objects are pairs $(A,n) \in \mathrm{Ob}(\mathbf{C}) \times \mathbb{Z}$, and the collection of arrows $(A,n) \to (B,m)$ corresponds to the colimit (in **Set**)

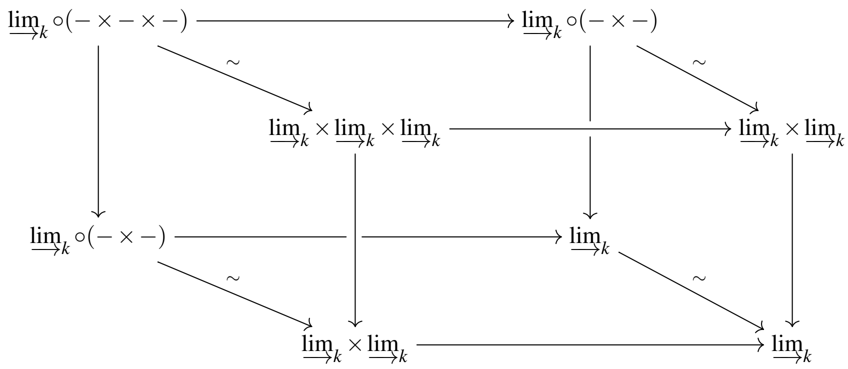
$$\varinjlim_{k \in \mathbb{N}} [\Omega^{n+k} A, \Omega^{m+k} B]_{\mathbf{C}}.$$

This is a well posed construction (or in other words $\mathrm{SHo}(\mathbf{C})$ really defines a category), because there exists a canonical arrow

$$\begin{array}{ccc}
\varinjlim_{k\in\mathbb{N}}[\Omega^{n+k}A,\Omega^{m+k}B]_{\mathbf{C}} \times \varinjlim_{k\in\mathbb{N}}[\Omega^{m+k}B,\Omega^{r+k}C]_{\mathbf{C}} & & \\
(\star)\big\downarrow\wr & & \\
\varinjlim_{k\in\mathbb{N}}\Big([\Omega^{n+k}A,\Omega^{m+k}B]_{\mathbf{C}} \times [\Omega^{m+k}B,\Omega^{r+k}C]_{\mathbf{C}}\Big) & \xrightarrow[\varinjlim \Omega^k c_{ABC}]{} & \varinjlim_{k\in\mathbb{N}}[\Omega^{n+k}A,\Omega^{r+k}C]_{\mathbf{C}}
\end{array}$$

induced by functoriality of $\varinjlim_k$ (isomorphism $(\star)$ follows from **[Schapira]**, Theorem **3.1.6**).

Associativity of the composition map follows from associativity of c_{ABC} (one has to check that a square is commutative, and this square is precisely one of the vertical faces of the cube of natural transformations between functors



the arrows joining the two faces being isomorphisms).

Define a family of correspondences on objects of $\mathrm{Ho}(\mathbf{C})$, $\Omega^i\colon \mathrm{SHo}(\mathbf{C}) \to \mathrm{SHo}(\mathbf{C})$, as

$$\Omega^i\colon (A,n) \mapsto (A,n+i).$$

If all these Ω^i are functors one has the following results "for free":

- $\mathrm{SHo}(\mathbf{C})$ contains a faithful copy of $\mathrm{Ho}(\mathbf{C})$, obtained by $\Omega^0\colon A \mapsto (A,0)$ (all of them are embeddings, because they are equivalences: $\Omega^i \circ \Omega^j = \Omega^{i+j}$, hence $(\Omega^i)^{-1} = \Omega^{-i}$);

- The functor $\Omega^1\colon (A,0) \mapsto (A,1)$ plays the rôle of a *shift* functor.

Proof of the functoriality of $\{\Omega^i\}$. First of all elements of $\hom_{\mathrm{SHo}(\mathbf{C})}((A,n),(B,m))$ belong to

$$\Big(\textstyle\bigsqcup_k M_k\Big)\Big/\!\sim$$

(it is the explicit characterization of the colimit in study: we call $M_k = [\Omega^{n+k}A,\Omega^{m+k}B]_{\mathbf{C}}$ for short) where the relation $\sim$ is defined by $(f \in M_a) \sim (g \in M_b)$ if and only if there exists $c \geq \max\{a,b\}$ such that $\alpha_{ca}(f) = \alpha_{cb}(g)$, where $\alpha_{mn} \colon M_n \to M_n$ is defined for all $n \leq m$ by sending $f \mapsto \Omega^{m-n} f$.

It is evident that all this defines for any $k \in \mathbb{Z}$, and by functoriality of $\varinjlim_k$, two arrows

$$[\Omega^{n+k}A, \Omega^{m+k}B]_{\mathbf{C}} \xrightarrow{\Omega^i} [\Omega^{n+i+k}A, \Omega^{m+i+k}B]_{\mathbf{C}}$$

$$\varinjlim_{k\in\mathbb{N}} [\Omega^{n+k}A, \Omega^{m+k}B]_{\mathbf{C}} \xrightarrow{\varinjlim \Omega^i} \varinjlim_{k\in\mathbb{N}} [\Omega^{n+i+k}A, \Omega^{m+i+k}B]_{\mathbf{C}}$$

because if $n+i+\bar{k} < 0$, there always exists $K_{i,n} > \bar{k}$ such that $n+i+K_{i,n} > 0$, hence for any $\ell \geq K_{i,n}$ the set $[\Omega^{n+i+\ell}A, \Omega^{m+i+\ell}B]$ is well-defined, and

$$\varinjlim_{k\in\mathbb{N}} [\Omega^{n+i+k}A, \Omega^{m+i+k}B]_{\mathbf{C}} \cong \varinjlim_{\ell\geq K} [\Omega^{n+i+\ell}A, \Omega^{m+i+\ell}B]_{\mathbf{C}}$$

because it is the same colimit, computed precomposing a cofinal functor (namely the inclusion $\iota_K \colon \mathbb{N}_{\geq K} \hookrightarrow \mathbb{N}$). □

REMARK 4 : *The stable homotopy category is triangulated by the inverse of the loop functor $\Sigma = \Omega^{-1}$. Distinguished triangles are those of the form*

$$(\Omega B, n) \to (F, n) \to (E, n) \to (B, n)$$

where $E \twoheadrightarrow B$, $F \hookrightarrow E$ is the homotopy inclusion of the typical fibre and $\Omega B \to F$ can be obtained exploiting the action $F \times \Omega B \to F$, as the image of the basepoint under the adjoint of the natural action map, $F \to F^{\Omega B}$.

Refer to **[Holm]** to get acquainted with the powerful machinery of triangulated categories; the fact that the loop object of the codomain of a fibration acts on the fibers of this fibration is a far reaching generalization of the well-known homotopy-theoretic topological analogue where $\Omega B \cong \pi_1(B)$: see **[Brown]**, Propositions **3** and **4**.

A pointed cfo $\mathbf{C}$ is said to be *stable* if the loop functor Ω is already an autoequivalence of $\mathrm{Ho}(\mathbf{C})$. If $\mathbf{C}$ is a stable cfo, then $\Omega^0 \colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{SHo}(\mathbf{C})$ is an equivalence, hence $\mathrm{Ho}(\mathbf{C})$ is itself triangulated by the shift functor Ω^{-1}.

Triangulated categories are somehow the best approximation of an abelian category (i.e. of a place where the machinery of Homological Algebra applies): we are now interested in taking the axiomatic path proposed by Eilenberg and Steenrod (see **[Steenrod]** and **[Vick]** to a precise account about this approach). We intentionally light up a little the discussion, addressing the interested reader to deeper presentations as **[Rotman]**. The theory of abstract stable homotopy presented in **[Uuye]** is almost the same of that in the seminal paper by **[Heller]**.

3.1 Homology Theories.

We now collect in a single subsection various results linked to *axiomatic homology theory* in the sense of the following Definition.

Definition 3.2 (Homology theory)**.** A *homology theory* in a pointed cfo is a homological functor $\mathcal{H}\colon \mathrm{SHo}(\mathbf{C}) \to \mathbf{Ab}$.

Definition 3.3 ($\mathcal{H}$-equivalences and $\mathcal{H}$-acyclic objects)**.** A morphism $q\colon A \to B$ is said to be an $\mathcal{H}$-equivalence with respect to a homology theory $\mathcal{H}$ on $\mathbf{C}$ if

$$\mathcal{H}(\Omega^n t)\colon \mathcal{H}(A,n) \to \mathcal{H}(B,n)$$

is an isomorphism in $\mathbf{Ab}$ for any $n \in \mathbb{Z}$. An object X is said $\mathcal{H}$-acyclic if $\mathcal{H}(X,n) = 0$ for any $n \in \mathbb{Z}$. The class of $\mathcal{H}$-equivalences form a multiplicative system which is compatible with the triangulated structure (see **[Krause]**, §**4.3**).

Classical Homological Algebra suggests the existence of a link between the two last notions: indeed a morphism $q\colon A_\bullet \to B_\bullet$ in a triangulated category is a quasi-isomorphism (i.e. an H-equivalence, H being classical (co)homology) if and only if the triangle $A_\bullet \xrightarrow{q} B_\bullet \to 0_\bullet \to A_\bullet[1]$ is distinguished: it is now possible to prove that

REMARK 5 : *A morphism $f\colon A \to B$ is a $\mathcal{H}$-equivalence if and only if its homotopy fiber is $\mathcal{H}$-acyclic (this is precisely a corollay of the existence of a long exact sequence induced by a a distinguished triangle*

$$(\Omega B,n) \to (Ff,n) \to (A,n) \xrightarrow{\Omega^n f} (B,n)).$$

This has a useful corollary: a fibration with typical fibre F is an $\mathcal{H}$-equivalence if and only if F is $\mathcal{H}$-acyclic.

The class of $\mathcal{H}$-equivalences are the WK part of a cfo structure on $\mathbf{C}$ (fibrations are the same). The cfo structure is denoted $R_{\mathcal{H}}\mathbf{C}$.

REMARK 6 : *We can take a slightly more general path: call S-*equivalences *the arrows in the smallest multiplicative system S^{-1}WK compatible with the triangulation and containing $S \subseteq Mor(\mathbf{C})$ (this is precisely the collection of all $t\colon A \to B$ which are $\mathcal{H}$-equivalences with respect to* any *homology theory $\mathcal{H}$ for which any $s \in S$ is a $\mathcal{H}$-equivalence: it's easy to see that any $A \xrightarrow{f} B \xleftarrow{t} C$ can be completed into a square where if t lies in S^{-1}WK then the parallel arrow also lies in S^{-1}WK; this amounts to the existence of a right calculus of fractions).*

The classes S^{-1}WK and $\mathrm{FIB}_{\mathbf{C}}$ define a cfo-structure on $\mathbf{C}$, and the resulting cfo is pointed if $\mathbf{C}$ was. This structure is denoted $R_S\mathbf{C}$.

The stable homotopy category $\mathrm{SHo}(R_S\mathbf{C})$ is triangle-equivalent to the Verdier localization (see **[Krause]**, §**4.6**) $\mathrm{SHo}(\mathbf{C})[(\Omega^0 S)^{-1}]$.

4 THE CFO STRUCTURE ON Top.

Notation and base assumption. We denote **Top** the category of *Kelley spaces*, i.e. the category of *compactly generated Hausdorff spaces* (see **[Mac Lane]** and

[**Steenrod**]'s paper); this is the usual candidate for a "well behaved" category **Spaces** of topological spaces, which can be replaced with any other having a closed symmetric monoidal structure (i.e. the functor $X \times -$ admits a right adjoint $(-)^X$ for any $X \in \mathbf{Spaces}$)[3].

The aim of this section is to show that the category **Top** can be made into a cfo in a really elementary fashion: this will be the building block of the cfo structure we want to impose to C^***-Alg** (see next section).

Definition 4.1 (0-Fibration)**.** Define an arrow $p \in \hom_{\mathbf{Top}}(E,B)$ to be a 0-*fibration* if any commutative square

$$\begin{array}{ccc} \{0\} & \longrightarrow & E \\ {\scriptstyle j_0}\big\downarrow & \overset{\alpha}{\nearrow} & \big\downarrow{\scriptstyle p} \\ [0,1] & \longrightarrow & B \end{array}$$

in **Top** admits a *diagonal filler* $\alpha \in \hom_{\mathbf{Top}}([0,1],E)$ which turns the two triangles (j_0 being the inclusion $\{0\} \hookrightarrow [0,1]$) into commutative ones.

We say for short that a fibration has the *right lifting property* (RLP) with respect to the inclusion $\{0\} \hookrightarrow [0,1]$.

We denote the class of 0-fibrations as $\text{FIB}_{\mathbf{Top},0}$.

Definition 4.2 (Weak equivalence)**.** Define an arrow $f \in \hom_{\mathbf{Top}}(A,B)$ to be a weak equivalence if it induces a bijection at the level of the zero-th homotopy set, i.e. if $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is a *bijection* between (pointed) sets.

We denote the class of weak 0-equivalences as $\text{WK}_{\mathbf{Top},0}$.

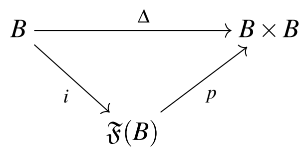
Theorem 4.1. The triple $(\mathbf{Top}, \text{WK}_{\mathbf{Top},0}, \text{FIB}_{\mathbf{Top},0})$ is a cfo, denoted π_0-**Top** for short.

Proof. For the sake of clarity let's enumerate what we have to prove:

- The classes of weak 0-equivalences and 0-fibrations are closed under composition;
- $\text{WK}_{\mathbf{Top},0}$ satisfies the 2-out-of-3 property.
- Any isomorphism is an acyclic fibration;
- The class of (acyclic) fibrations is closed under base change;
- For any $B \in \mathbf{Top}$ the unique map $B \to \{*\}$ is a fibration;

[3] A central point in the following discussion is to embed the category C^***-Alg** of C^*-algebras into the category of topological spaces; our procedure is not affected by the base assumption because any metric space (C^*-algebras are such) is obviously Hausdorff compactly generated.

- There exists a functor $\mathfrak{F}\colon \mathbf{Top} \to \mathbf{Top}$ such that the diagram



  commutes for any $B \in \mathbf{Top}$. The arrow i is the inverse of an acyclic fibration (in particular, $i \in \mathrm{WK}_{\mathbf{Top},0}$), and $p \in \mathrm{FIB}_{\mathbf{Top},0}$. The map Δ is the diagonal one: $b \mapsto (b,b)$.

The only two non-evident properties are the existence of enough (functorial) path objects and the closure of $\mathrm{FIB}_{\mathbf{Top},0} \cap \mathrm{WK}_{\mathbf{Top},0}$ under base change.

► Define $\mathfrak{F}(B) = \mathbf{Top}([0,1],B)$ ($= B^I$ in more suggestive notation): it is a topological space with respect the compact open topology a subbase of which is made by functions f sending a compact $K \Subset [0,1]$ into an open $V \in B$: thanks to the initial assumption the correspondence $B \mapsto B^I$ is functorial.

  To show that this defines a true path-object for B consider the canonical map $i\colon B \to \mathfrak{F}(B)$ sending b in the constant path $\gamma_b(t) \equiv b$. This is a continuous function thanks to the assumptions made on the topology of B^I: $\hom(B \times I, B) = \hom(B, \mathbf{Top}([0,1],B))$.

  The map i is indeed a section of the morphism of evaluation at $0 \in [0,1]$, $\mathrm{ev}_0\colon \gamma \mapsto \gamma(0)$, which is again continuous thanks to the assumptions made on our category of spaces. The map ev_0 is a fibration, because if we are given a path $\gamma\colon [0,1] \to B$, we can define $\widetilde{\gamma}\colon [0,1] \to \hom([0,1],B)$ by $\widetilde{\gamma}(t) = \gamma_t$, where $\gamma_t(s) = \gamma(t + (1-t)s)$. This is easily seen to be a continuous map again by adjoint nonsense, or if you want just because it is exactly the map which corresponds to $[0,1] \times [0,1] \to B\colon (t,s) \mapsto \gamma(t + (1-t)s)$ under the bijection

$$\hom_{\mathbf{Top}}([0,1] \times [0,1], B) \cong \hom_{\mathbf{Top}}([0,1], B^I).$$

  Finally, ev_0 is easily shown to be (much more than) 0-acyclic: it is an homotopy equivalence (in the classical sense of inducing isomorphisms between *all* homotopy groups). Indeed, if we define $\Phi\colon B^I \times I \to B^I$ in such a way that $\Phi(\gamma,t) = {}_t\gamma$, ${}_t\gamma(s) = \gamma((1-t)s)$, this map is continuous and realizes B^I as a deformation retract of $i(B)$.

  It remains only to define the map p: just send γ in $p(\gamma) = (\gamma(0), \gamma(1))$ in such a way that $(p \circ i)(x) = p(i(x)) = (x,x) = \Delta(x)$, and notice that p is a fibration because it is a product of fibrations.

  More geometrically, suppose that a distinguished point (i.e. a path γ) in B^I is fixed. Its image under p is $(x_0, y_0) = (\gamma(0), \gamma(1))$. Suppose we are given a path in $B \times B$ starting in (x_0, y_0): it is obvious that this amounts to give *two* paths, $\alpha, \beta\colon [0,1] \to B$ such that $\alpha(0) = x_0$ and $\beta(0) = y_0$.

Now for any $t \in [0,1]$ define $\alpha_t, \beta_t\colon I \to B$ by $\alpha_t(s) = \alpha(ts)$, $\beta_t(s) = \beta(ts)$ (they are the paths α, β restricted to $[0,t]$), and

$$\Psi\colon [0,1] \to B^I : \Psi(t) = \overline{\alpha_t} \star \gamma \star \beta_t,$$

where the $\star$ denotes junction of paths and $\overline{\alpha_t}$ is the path in the inverse direction (you go through $\overline{\alpha} \star \gamma \star \beta$ more and more until you tread it all: this is the desired lifting because it's easy to see that $p(\alpha \star \cdots \star \beta) = (\alpha(0), \beta(1))$).

- The class of fibrations is closed under base change: consider the diagram

$$\begin{array}{ccccc}
\{0\} & \longrightarrow & A \times_B E & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow\!\!\downarrow p \\
[0,1] & \xrightarrow[\gamma]{} & A & \xrightarrow[u]{} & B;
\end{array}$$

if we apply RLP to the outer rectangle, we find $\alpha\colon [0,1] \to E$ lifting $u \circ \gamma$, and the universal property of the pullback implies the existence of a unique $\beta\colon [0,1] \to A$ lifting γ.

Lemma 4.1. Acyclic fibrations $p\colon E \xrightarrow{\sim} B$ are surjective maps.

Proof. If $\pi_0(p)$ is surjective, then any arcwise connected component in B must be reached by a continuous arc, hence for any $\mathcal{C} \in \pi_0(B)$ there exists $\mathcal{D}$ such that $\mathcal{C} = \pi_0(p)(\mathcal{D})$, i.e. for any $b \in [b] = \mathcal{C} \subset B$ there exists an $e \in [e] = \mathcal{D} \subset E$ such that $[p(e)] = b$, which is equivalent to say that for any $b \in B$ there exists a continuous path between b e $p(e)$ for some $e \in E$.

Now, given a point $b \in B$ let's find a path $h\colon b \rightsquigarrow p(e')$ for some $e' \in E$; lift h to a path $\alpha\colon [0,1] \to E$, which must have endpoint $\alpha(1)$ such that $p(\alpha(1)) = b$:

$$\begin{array}{ccc}
\{0\} & \longrightarrow & E \\
\downarrow & \overset{\alpha}{\nearrow} & \downarrow p \\
[0,1] & \xrightarrow[h]{} & B.
\end{array}$$

□

Now let's consider the diagram induced by $\pi_0(-)$:

$$\begin{array}{ccccc}
* & \longrightarrow & \pi_0(A \times_B E) & \longrightarrow & \pi_0(E) \\
\downarrow & & \downarrow & & \sim \downarrow \pi_0(p) \\
* & \longrightarrow & \pi_0(A) & \longrightarrow & \pi_0(B).
\end{array}$$

Pulling back a surjective map returns a surjective map in **Top**, because of the exactness properties of this category (it is an elementary topos, so it is Barr-exact), and the functor π_0 preserves surjections (it's easily seen). Injectivity of the pulled-back map can be showed directly: suppose $[f(x)] = [f(y)]$, so there exists a continuous arc $\gamma\colon fx \leftrightsquigarrow fy$, and a continuous arc starting at x in $A \times_B E$, say α, such that $f(\alpha(t)) = \gamma$: this is responsible of $[x] = [y]$. □

5 Homotopy Theory in C^*-Alg.

A few categorical properties of C^*-Alg. Recall that a *topological (complex) vector space* $(V_\mathbb{C}, \tau, +)$ is an internal $\mathbb{C}$-module in our category **Top** of "nice" topological spaces; a *Banach space* $B = (\langle V_\mathbb{C}, +\rangle, \|\cdot\|)$ is a normed[4] topological vector space, whose topology is induced by that norm, and which is (Cauchy-) complete with respect to the metric induced by the same norm. A *Banach algebra* $\mathfrak{A} = (\langle A, +, \cdot\rangle, \|\cdot\|)$ is a Banach space endowed with a bilinear multiplication turning it into a $\mathbb{C}$-algebra, such that $\|a \cdot b\| \le \|a\|\|b\|$; this entails that the multiplication $V \times V \to V$ is automatically (and separately in both variables) continuous. Finally, a (Banach-)*-algebra $\mathscr{A} = (\mathfrak{A}, (-)^*)$ is a Banach algebra endowed with an involutory conjugate-linear anti-automorphism $(-)^*\colon A \to A$.

Let $\mathbf{Vect}_\mathbb{C}, \mathbf{Ban}, \mathbf{BanAlg}, {}^*\textbf{-Alg}$ respectively denote the categories of (topological) complex vector spaces, Banach complex spaces and bounded linear maps, Banach complex algebras and continuous algebra-homomorphisms, and complex *-algebras and algebra homomorphisms $f\colon \mathscr{A} \to \mathscr{B}$ such that $f\colon A \to B$ commutes with $(-)^*$: $f(a^*) = f(a)^*$; there is the chain of categorical inclusions

$$^*\textbf{-Alg} \subset \mathbf{BanAlg}_\mathbb{C} \subset \mathbf{Ban}_\mathbb{C} \subset \mathbf{Vect}_\mathbb{C}.$$

Define the category C^***-Alg** to be the full subcategory of those *-algebras satisfying the *-*property* of the norm:

$$\|a^* \cdot a\| = \|a\|^2$$

[4] The concept of a *seminorm* on a vector space A, $\|\cdot\|\colon A \to \mathbb{R}_+$ can be internalized exploiting two diagrams of sets and functions:

$$\begin{array}{ccc} \mathbb{C}\times A & \xrightarrow{\alpha_A} & A \\ {\scriptstyle |\cdot|\times\|\cdot\|}\downarrow & & \downarrow{\scriptstyle \|\cdot\|} \\ \mathbb{R}_+\times\mathbb{R}_+ & \xrightarrow[\mu_{\mathbb{R}_+}]{} & \mathbb{R}_+ \end{array} \qquad \begin{array}{ccc} A\times A & \xrightarrow{+_A} & A \\ {\scriptstyle \|\cdot\|\times\|\cdot\|}\downarrow & & \downarrow{\scriptstyle \|\cdot\|} \\ \mathbb{R}_+\times\mathbb{R}_+ & \xrightarrow[+_{\mathbb{R}}]{} & \mathbb{R}_+ \end{array}$$

the first is asked to be commutative, and in the second the composition $\|\cdot\| \circ +_A$ is asked to be less or equal to $+_\mathbb{R} \circ (\|\cdot\| \times \|\cdot\|)$ in the obvious partial order in $\hom(A \times A, \mathbb{R}_+)$, $f \le g \iff f(a,b) \le g(a,b)$ for all $(a,b) \in A \times B$. A *norm* on A is a seminorm $\|\cdot\|\colon A \to \mathbb{R}_+$ such that the composition $\mathbb{C} \xrightarrow{0_A} A \xrightarrow{\|\cdot\|} \mathbb{R}_+$ is the zero-arrow $\mathbb{C} \to \mathbb{R}_+$.

(in other words the composition $a \mapsto (a,a) \mapsto (a^*,a) \mapsto (a^* \cdot a \mapsto \|a^* \cdot a\|$ equals $a \mapsto \|a\| \mapsto \|a\|^2$).

C^***-Alg** is a complete and cocomplete, **Top**-concrete and **Top**-enriched category, the functor C^***-Alg** $\to$ **Top** being the forgetful one, regarding a C^*-algebra $\mathscr{A} = (\mathfrak{A}, (-)^*)$ (or the support A of the whole structure, for short) as a compactly generated space.

In this way, giving $\hom_{\mathrm{C}^*}(A,B)$ the subspace topology via the inclusion in $\hom_{\mathbf{Top}}(A,B)$, Gel'fand duality becomes an equivalence of *enriched categories* (see **[Borceux]**, **II.6.7**).

Lemma 5.1. The main reason we repeatedly underlined the assumption of cartesian closure for our base category **Spaces** is the that we can obtain the following canonical identification:

$$\hom_{\mathbf{Top}}(X, \hom_{\mathrm{C}^*}(A,B)) \cong \hom_{\mathrm{C}^*}(A, C(X) \otimes B) \tag{1}$$

where $A,B \in \mathrm{C}^*$**-Alg**, and X is a compact Hausdorff space.

Proof. It is a straightforward example of adjoint-nonsense, based on the classical fact that the function space $B^X = \mathbf{Top}(X,B)$ can be given the structure of a C^*-algebra by

$$\|f\| := \sup_{x \in X} \|f(x)\|_B ,$$

and this C^*-algebra is *-isomorphic to $C(X) \otimes B$. Hence

$$\begin{aligned} \hom_{\mathbf{Top}}(X, B^A) &\cong \hom_{\mathbf{Top}}(X \times A, B) \\ &\cong \hom_{\mathbf{Top}}(A \times X, B) \\ &\cong \hom_{\mathbf{Top}}(A, B^X) \\ &\cong \hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(A, C(X) \otimes B). \end{aligned}$$ □

One can now define a category $\pi_0\mathrm{C}^*$**-Alg** starting from C^***-Alg** with the same objects as C^***-Alg** and

$$\hom_{\pi_0\mathrm{C}^*\text{-}\mathbf{Alg}}(A,B) := \pi_0\big(\hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(A,B)\big)$$

(this can be motivated by the fact that if $X \in \mathbf{LCTop}$, homotopy classes of continuous maps $X \to Y$ in **Top** can be identified with arcwise connected components of the map space Y^X in the compact-open topology.

Definition 5.1 (Weak C^*-equivalence)**.** Define a *-morphism to be a *weak equivalence* if for any $D \in \mathrm{C}^*$**-Alg** the induced map

$$t_\#\colon \hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(D,A) \to \hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(D,B)$$

is a weak equivalence in π_0-**Top**. (Notice that $t \in \mathrm{Mor}(\mathrm{C}^*$**-Alg**$)$ is a weak equivalence if and only if $\pi_0(t) \in \pi_0\mathrm{C}^*$**-Alg** is an invertible map.)

The class of weak equivalences in C^***-Alg** is denoted $\mathrm{W}\kappa_{\mathrm{C}^*}$.

Definition 5.2 (C^*-fibration)**.** Define a *-morphism $p\colon E \to B$ to be a *Schochet fibration* if for any $D \in C^*\textbf{-Alg}$ the induced map

$$p_\#\colon \hom_{C^*\textbf{-Alg}}(D,E) \to \hom_{C^*\textbf{-Alg}}(D,B)$$

is a fibration in π_0-**Top**.

The class of Schochet fibrations in C^***-Alg** is denoted FIB_{C^*}.

Notation. In [**Schochet**]'s paper our π_0-fibrations are called π_0-*co*fibrations, because the RLP asked to one of our π_0-fibrations corresponds to the LLP asked to $\operatorname{Spec} B \to \operatorname{Spec} A$ to be a cofibration in **Top** (**LCHaus** $\subset$ **Top** in our notations), via Gel'fand duality.

Similarly, one can show that the suspension functor $\Sigma\colon \mathbb{S}^1 \wedge -$ corresponds via Gel'fand duality to the functor $C_*(\mathbb{S}^1) \otimes -$; proving this boils down to the chain of natural isomorphisms (the first of which is Lemma **5.1**)

$$\begin{aligned} B^{\mathbb{S}^1} &= \hom_{\textbf{Top}_*}(\mathbb{S}^1, B) \\ &\cong C_*(\mathbb{S}^1) \otimes B \\ &\cong C_*(\mathbb{S}^1) \otimes C_*(\operatorname{Spec} B) \\ &\cong C_*(\mathbb{S}^1 \wedge \operatorname{Spec} B) \end{aligned}$$

where $C_*(X)$ is the algebra of functions $(X, \mathrm{pt}_X) \to (\mathbb{C}, 0)$. This suggests to define the path object in C^***-Alg** is such a way that it corresponds to the path object $(\operatorname{Spec} B)^{[0,1]}$ in **Top**: it is the algebra $C_*([0,1]) \otimes B$, clearly a functorial correspondence.

Theorem 5.1 ([**Uuye**], thm. 2.11)**.** The triple $(C^*\textbf{-Alg}, \text{WK}_{C^*}, \text{FIB}_{C^*})$ is a pointed cfo. The homotopy category $\text{Ho}(\mathbf{C})$ is the category $\pi_0 C^*\textbf{-Alg}$ defined before, hence we denote the triple $(C^*\textbf{-Alg}, \text{WK}_{C^*}, \text{FIB}_{C^*})$ as $\pi_0 C^*\textbf{-Alg}$ (we also denote this particular cfo structure on C^***-Alg** as the "π_0 structure").

Proof. The only thing we can't derive from Theorem **4.1** is that any Schochet fibration is a surjective map, and this follows easily "mimicking" the simple proof of Lemma **4.1** (see [**Uuye**], Prop. **2.10**). □

5.1 A Brief Interlude: Model Categories.

A model category is in some sense a *smoothing* of the notion of cfo. In a few words, a model category consists of a cfo $(\mathbf{C}, \text{WK}, \text{FIB})$ endowed with an additional class of arrows COF, the elements of which are called *cofibrations*, having suitable stability and lifting properties with respect to fibrations. These properties give a reasonably general context in which it is possible to set up the basic machinery of Homotopy Theory.

Definition 5.3 (Model Category)**.** A *(Quillen) model category* is a small-complete and small-cocomplete category endowed with three distinguished classes of morphisms: *weak equivalences*, Wk; *fibrations*, Fib; *cofibrations*, Cof, such that the following axioms are satisfied:

- $(\mathbf{C}, \textsc{Wk})$ is a category with weak equivalences;
- $\textsc{Wk}, \textsc{Fib}, \textsc{Cof}$ are stable under taking retracts;
- For any commutative square

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & W, \end{array}$$

  where $i \in \textsc{Cof}$ and $p \in \textsc{Fib}$, If either i or p is acyclic, then there exists a lifting $Y \to Z$. In other words, acyclic fibrations/cofibrations have the *right/left lifting property* (RLP, LLP for short) with respect to fibrations/cofibrations;

- $(\textsc{Wk} \cap \textsc{Fib}, \textsc{Cof})$, $(\textsc{Fib}, \textsc{Wk} \cap \textsc{Cof})$ are (weak) factorization systems in $\mathbf{C}$.

REMARK 7 : *Mutual lifting properties are what really define the notion of model category: a model category is uniquely determined by the datum of weak equivalences and fibration* or *by the datum of weak equivalences and cofibrations: in the first case, cofibrations are maps having the* LLP *with respect to acyclic fibrations, and in the second case fibrations are maps having the* RLP *with respect to acyclic cofibrations (see Prop.* ***3.13*** *in* ***[Dwyer-Spalinski]****).*

Examples of model categories live in algebraic, topological and even pure-categorical contexts. Refer again to **[Dwyer-Spalinski]** to have plenty of examples and explicit constructions: "each of these settings has its own technical and computational peculiarities [in general the task of proving that a particular choice of weak equivalences and (co)fibrations really gives a model category is extremely long and involved: see for example **[Gelfand-Manin]**, **V.1.2-V.2.4**], but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language.

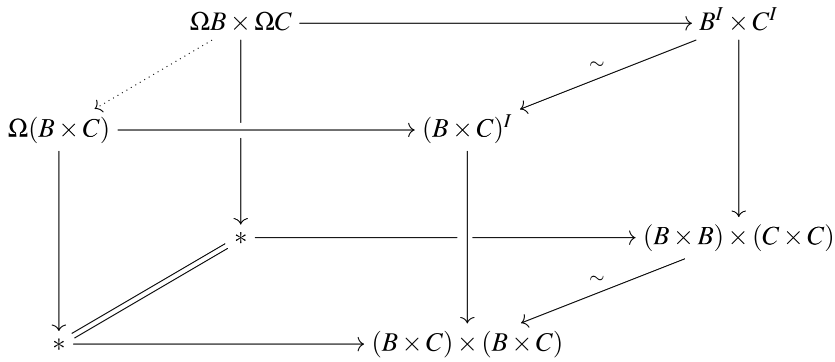
What is the *suspension* of an augmented commutative algebra? One of incidental appeals of Quillen's theory (to a topologist!) is that it both makes a question like this respectable and gives it an interesting answer."

REMARK 8 [[UUYE], EXAMPLE **1.4**]: *If* $(\boldsymbol{C}, \textsc{Wk}, \textsc{Fib}, \textsc{Cof})$ *is a model category, then the full subcategory* $\boldsymbol{C}_{\mathrm{fib}}$ *consisting of the sole fibrant objects in* $\boldsymbol{C}$ *is a cfo, by restricting the weak equivalences and the fibrations to* $\boldsymbol{C}_{\mathrm{fib}}$*.*

5.2 π_0C*-**Alg** is not a rhm.

Proposition 5.1. The loop functor $\Omega\colon \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \to \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg})$ preserves finite products.

Proof. It is a one-diagram proof: consider the cube



and notice that by functoriality the arrow $\Omega B \times \Omega C \to \Omega(B \times C)$ (which is the canonically induced map) must be an isomorphism. □

If we were able to prove that Ω doesn't commute with *infinite* products we would be able to show that[5] Ω isn't part of an adjoint pair $\Sigma \dashv \Omega$; this should seem quite easy to check, but notice (see for example the first pages of **[Warner]** and the paper by **[Avitzour]**) that the construction of arbitrary products and coproducts in C*-**Alg** if very far from explicit.

So we choose to take the direct path. This will imply as a corollary that the π_0 structure built in Theorem **5.1** doesn't come from a model structure in the sense of Quillen on C*-**Alg**, i.e. that there is no model structure **M** on C*-**Alg** such that $\pi_0\mathrm{C}^*\textbf{-Alg} = \mathbf{M}_{\mathrm{fib}}$ (we call these structures *restricted homotopy models*, rhm for short). Indeed, recall that if **C** is a model category, then the full subcategory $\mathbf{C}_{\mathrm{fib}}$ of its fibrant objects becomes in a natural way a cfo (this is Remark **8**). The natural inclusion "passes to homotopy", and the loop functor admits a restriction $\Omega|_{\mathrm{F}}\colon \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}}) \to \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}})$ and a left adjoint, the *suspension functor* $\Sigma\colon \mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{C})$.

Now, the category C*-**Alg** of commutative C*-algebras is a reflective subcategory of C*-**Alg**, let's call it C*-**Alg**$_c$ (the reflector is the *abelianization* functor $A \mapsto A^{\mathrm{ab}}$). The (2-)functor Ho is particularly well-behaved with respect to this reflection (more technically, the inclusion functor is part of a *Quillen adjunction*), which ensures that the reflection $\mathrm{C}^*\textbf{-Alg} \leftrightarrows \mathrm{C}^*\textbf{-Alg}_c$ descends to an adjunction

[5] This is because in a suitably "smooth" category as **Top** is, a functor is a right adjoint if and only if it commutes with limits; the rather technical condition which ensures this is Freyd's SAFT, see [**Mac Lane**].

$\mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \leftrightarrows \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c)$ which is again a reflection. The diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) & \longrightarrow & \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \\ \uparrow & & \uparrow \\ \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c) & \xrightarrow[\Omega|_{\mathrm{C}^*\textbf{-Alg}_c}]{} & \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c) \end{array}$$

commutes. From this we can deduce that the stable homotopy category of $\mathrm{C}^*\textbf{-Alg}_c$ is a full triangulated subcategory of $\mathrm{SHo}(\mathrm{C}^*\textbf{-Alg})$; now Theorems **I.1.1** and **I.2.2** in **[Quillen]** say that if $\mathbf{C}$ is a model category, the loop functor $\Omega|_c \colon \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}}) \to \mathrm{Ho}(\mathbf{C}_{\mathrm{fib}})$ must be a right adjoint.

Theorem 5.2. The loop functor $\Omega \colon \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \to \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg})$ doesn't admit a left adjoint, so the cfo structure on $\mathrm{C}^*\textbf{-Alg}$ isn't the restriction of a Quillen model structure on $\mathrm{C}^*\textbf{-Alg}$.

Proof. Our plan is to prove that the loop object functor $\Omega \colon \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c) \to \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c)$ doesn't admit a left adjoint: this will imply that $\Omega \colon \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \to \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg})$ doesn't admit it too, because if there wa a functor Σ such that $[\Sigma A, B] \cong [A, \Omega B]$, then the composition

$$\mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c) \xrightarrow{\iota} \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \xrightarrow{\Sigma} \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}) \xrightarrow{(-)^{\mathrm{ab}}} \mathrm{Ho}(\mathrm{C}^*\textbf{-Alg}_c)$$

would be a left adjoint to $\Omega|_c$.

Now that the plan of the proof is clear, notice that by Gel'fand-Naimark duality it suffices to show that the (topological) suspension functor $\Sigma \colon \mathbf{CHaus}_* \to \mathbf{CHaus}_*$ doesn't admit a *right* adjoint: it is a general consequence of the presence of an antiequivalence $\mathbf{C}^{\mathrm{op}} \cong \mathbf{D}$ that the adjunction $F \colon \mathbf{C} \leftrightarrows \mathbf{C} \colon G$ transports to an adjunction $\mathbf{D} \leftrightarrows \mathbf{D}$ (a suitable diagram must commute: contravariant nonsense does the rest).

> In the end the whole proof boils down to show that $\Sigma \colon X \mapsto X \wedge \mathbb{S}^1$ doesn't admit a right adjoint.

In particular let's show that the functor $A \mapsto [\Sigma A, \mathbb{S}^1]$ is not representable. Suppose it is and hope to bump into a contradiction: there exists a compact Hausdorff space $Y \in \mathrm{Ho}(\mathbf{CHaus}_*)$ such that

$$[\Sigma A, \mathbb{S}^1] \cong [A, Y].$$

Now, in the whole category $\mathbf{Top}_*$ of pointed spaces the functor Σ *does admit* a right adjoint (precisely the loop-space functor $\Omega \colon (X,*) \mapsto (X,*)^{(\mathbb{S}^1,1)} \cong \pi_1(X,*)$) , hence we have the chain of natural bijections

$$\begin{aligned} [A,Y]_{\mathbf{Top}_*} &\cong [A,Y]_{\mathbf{CHaus}_*} \\ &\cong [\Sigma A, \mathbb{S}^1]_{\mathbf{CHaus}_*} \\ &\cong [A, \Omega \mathbb{S}^1]_{\mathbf{Top}_*} \end{aligned}$$

Yoneda lemma entails now that the natural bijection we found for any X *must come* from an isomorphism $f\colon Y \to \Omega\mathbb{S}^1$ in $\mathrm{Ho}(\mathbf{CHaus}_*)$, namely from a weak (homotopy) equivalence $Y \to \Omega\mathbb{S}^1$, inducing degree-wise isomorphisms between homotopy groups $\pi_i(Y)$ and $\pi_i(\Omega\mathbb{S}^1)$. The space $\mathbb{S}^1$ being a $K(\mathbb{Z},1)$, this boils down to say that $\pi_i(Y) \cong \pi_{i+1}(\mathbb{S}^1)$ for $i = 0, 1$.

Now $\pi_0(Y) \cong \pi_0(\Omega\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$, hence Y must have an infinite number of arcwise connected components.

Let's see why this is the desider contradiction (and notice that we can't conclude now the proof because there is plenty of compact spaces having an infinite number of *non-open* connected components).

In particular, let's show that any $\mathcal{C} \in \pi_0(Y)$ must be an open set: in the composition

$$Y \xrightarrow{\sim} \Omega S^1 \xrightarrow{\deg} \mathbb{Z}$$

the function deg is an isomorphism between topological groups, once we endowed the codomain with the discrete topology, hence the preimage of open sets in $\mathbb{Z}$ (=singletons) must be open, and is exactly one of the arcwise connected components of Y. □

A Appendix

A.1 The π_n-structure on Top.

This paragraph is adressed to answering some somehow natural questions stemmed from Theorem **4.1**:

1. Is π_0-**Top** the *degree-zero element* of a family $\{\pi_n\text{-}\mathbf{Top}\}_{n\in\mathbb{N}}$ of fibrant structures on **Top**, recognizing as weak equivalences as a map $f\colon X \to Y$ inducing isomorphisms between all homotopy groups/pointed spaces $\pi_k(X,x) \cong \pi_k(Y, f(x))$ for any $0 \le k \le n$, and for any choice of the base-point?

2. Can a suitable (2-)"limiting" procedure as $\varinjlim_n \big(\pi_n\text{-}\mathbf{Top}\big)$ of these fibrant structures recover the fibrant structure induced forgetting cofibrations and mutual lifting properties of a suitable model structure on **Top** (the limit has to be understood in the (2-)category **ModCat** of model categories, whose 1-cells are Quillen pairs)[6]?

3. Is this more general path useful in application to C*-algebra theory? For example, a positive answer to question 1 would give a countable family of

[6]Morphisms between model categories are *pairs* of adjoint functors $\mathcal{F}\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon \mathcal{G}$: in fact, the natural request for a functor between model categories is that it induces a functor between the homotopy categories $\mathrm{Ho}(\mathbf{C}) \to \mathrm{Ho}(\mathbf{D})$, and to this end the very least we can ask is that the left adjoint preserves cofibrations and acyclic cofibrations, or (it is equivalent) the rght adjoint preserves fibrations and acyclic fibrations, or again that the left adjoint preserves (acyclic) cofibrations, and the right adjoint preserves (acyclic) fibrations.

rhms on **Top**, inducing a similar one on C*-**Alg** by "transport of structure". Now the question is: can the cfo structure on C*-**Alg** induced by a rhm on **Top** be itself a rhm on C*-**Alg**? Maybe the same question can be asked in a more general flavour:

> Let **M** be a symmetric monoidal closed model category, and consider the (2-)category **M**-**Cat** of **M**-enriched categories. Given a **M**-category **C**, can it become a model category defining fibrations/weak equivalencess as maps $X \to Y$ such that $\hom_{\mathbf{C}}(-,X) \to \hom_{\mathbf{C}}(-,Y)$ is a fibration/weak equivalence (better to say: for any $A \in \mathbf{C}$ $\hom_{\mathbf{C}}(A,X) \to \hom_{\mathbf{C}}(A,Y)$ is a fibration/weak equivalence)? Can this become a (2-)functor **M**-**Cat** $\to$ **ModCat**?

Even if the condition for a map to be a π_n-equivalence seems rather obvious, the condition for a map to be a π_n-fibration has to be taken with care once one notices that it's not at all clear how the lifting condition with respect to paths given the initial point has to be extended. At first glance, it resembles a "truncation" of the condition for a map to be a Serre fibration: a private communication **[Uuye2]** with O. Uuye unfortunately pointed out that this is not the case: the pullback of a Serre (or even Hurewicz) acyclic fibration is not necessarily a π_n-equivalence.

For example, the Hopf fibration $S^3 \to S^2$ is a π_1-equivalence, but the pullback along $* \to S^2$ is $S^1 \to *$, which is not a π_1-equivalence. So the notion of a π_n-fibration has to be something stronger (not weaker) than the notion of a Serre fibration, which is not desirable.

As a minor issue, the definition of π_n-equivalence is not satisfactory for another reason: it is different from the *standard* definition of n-equivalence, where the maps $\pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ (at the highest level of homotopy groups) are only required to be surjective.

This stronger notion of n-fibration is provided by **[Donazar]**'s paper[7], where an n-fibration $p\colon E \to B$ in **Top** (or in a suitable cartesian closed subcategory of spaces) is defined to be a map having the RLP with respect to $V^{k-1} \to [0,1]^k$ for any $0 < k \le n+1$, *and* with respect to $V^{n+1} \to \partial[0,1]^{n+2}$, where V^{k-1} denotes the union of all faces of $[0,1]^k$ except for $[0,1]^{k-1} \times \{1\}$ in the case $n = 0$, this boils down to ask the RLP which defines 0-fibrations *plus* the RLP with respect to the inclusion $\sqcup \hookrightarrow \square$ of three sides of a square in a square ("any loop on the base can be lifted, given a piece of the path"):

$$\begin{array}{ccc} \sqcup & \longrightarrow & E \\ \downarrow & \overset{\alpha}{\nearrow} & \downarrow p \\ \square & \longrightarrow & B. \end{array}$$

[7] I'm thankful to D. White who found out this paper at http://mathoverflow.net/questions/112069/a-fibrant-objects-structure-on-top/114916

The proof of the main theorem in **[Donazar]**'s paper proceeds in parallel with the category $\mathbf{SSet} = \mathbf{Sets}^{\Delta^{op}}$ of simplicial sets, betraying a particularly *combinatorial* nature (the condition for a map to be a fibration is not so far from the definition of a Kan fibration in **SSet**); the link between this combinatorial approach and the third point remain at the moment untouched, but in view of **[Dell'Ambrogio,2]** and **[Østvær]**'s papers it seems that the well-known homotopical structure(s) on **SSet** can help in establishing analogous homotopical structure(s) on C^***-Alg**.

A.2 Cubical C^*-algebras.

The main idea in **[Østvær]**'s paper is to combine C^*-theory with the well-established homotopy theory of *cubical sets*; in the same way a simplicial set can be thought as a graded set $\{K_n\}$ with suitable functions $\partial_j^n \colon K_n \to K_{n-1}$ (*faces*), $s_j^n \colon K_n \to K_{n+1}$ (*degeneracies*) satisfying suitable *simplicial identities*[8], a cubical set can be characterized as a graded set $\{K_n\}$ with suitable functions $d_{i,\alpha}^n \colon K_n \to K_{n-1}$, $s_j^n \colon K_n \to K_{n+1}$ satisfying *cubical identities*:

$$d_{i,\alpha}^n \circ s_j^n = \begin{cases} s_{j-1}^n \circ d_{i,\alpha}^n & i < j \\ 1 & i = j \\ s_j^n \circ d_{i-1,\alpha}^n & i > j. \end{cases}$$

The category of cubical sets is again a presheaf category: defining $\boxdot$ to be the category having as objects the posets $[n] = \mathcal{P}(\{1,\dots,n\})$, for every $n \in \mathbb{N}$, and as arrows $[n] \to [m]$ monotone mappings which can be written as compositions of faces and degeneracies, a cubical set now is a functor $\square^{op} \to \mathbf{Sets}$. **[Østvær]** denotes the category of cubical sets with $\boxdot$**Sets**.

Now, if we consider the classical Yoneda embedding $\mathrm{yon}\colon \mathrm{C}^*\text{-}\mathbf{Alg}^{op} \to \mathbf{Sets}^{\mathrm{C}^*\text{-}\mathbf{Alg}} \colon A \mapsto \hom_{\mathrm{C}^*\text{-}\mathbf{Alg}}(A,-)$ we can find a faithful copy of C^***-Alg** in its (co)presheaf category, via a continuous functor yon. Now one can consider the subcategory of *cubical set-valued presheaves* $\mathrm{Fun}(\mathrm{C}^*\text{-}\mathbf{Alg}, \boxdot\mathbf{Sets}) \cong \mathrm{Fun}(\mathrm{C}^*\text{-}\mathbf{Alg}, \mathbf{Sets})^{\boxdot^{op}}$, obtaining the category of *cubical* C^**-spaces*. Mimicking the construction of the Reedy model structure[9] on a functor category $\mathrm{Fun}(\mathbf{C},\mathbf{D})$ one is lead to define a weak equivalence $\mathcal{X} \to \mathcal{Y}$ between cubical C^*-spaces as an objectwise weak equivalence $\mathcal{X}(A) \to \mathcal{Y}(A)$ between cubical sets.

A.3 C^*-categories and simplicial algebras.

[Dell'Ambrogio,2]'s paper defines a *cofibrantly generated*, $\boldsymbol{SSet}_{Quil}$*-enriched* monoidal model structure on the category C^***-Cat** of C^**-categories*:

[8] These identities can be deduced from the fact that a simplicial set is nothing but a presheaf over $\Delta = \mathbf{FinOrd}$, the category of totally ordered finite sets and monotone maps: see **[Mac Lane]**, §**7.5**. The category Δ is now *generated* by face and degeneracy arrows.

[9] http://ncatlab.org/nlab/show/Reedy+model+structure

- A C*-*category* is, roughly speaking, a category **C** enriched over the symmetric monoidal category of (complex) C*-algebras, such that for any $X \in \mathbf{C}$ the set $\hom_{\mathbf{C}}(X,X)$ is a *unital* C*-algebra: see **[Warner]**, ch. **15**. The class of all C*-categories becomes a (2-)category if we define 1-cells $\mathbf{C} \to \mathbf{D}$ to be the collection of all *-*functors* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$, and 2-cells to be *bounded* natural transformations $\mathcal{F} \to \mathcal{G}$ (see again **[Warner]**, ch. **15**).

- A *cofibrantly generated* model category consists of a model category in which acyclic cofibrations $\mathrm{COF} \cap \mathrm{WK}$ and fibrations FIB can be recovered as maps having the right lifting properties with respect to all maps in suitable *sets* I, J:

$$\mathrm{COF} \cap \mathrm{WK} = \mathrm{rlp}(I), \qquad \mathrm{FIB} = \mathrm{rlp}(J)$$

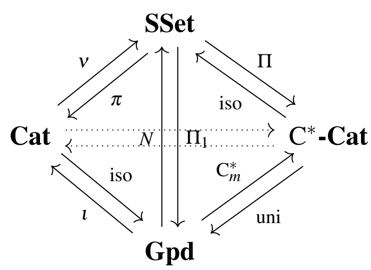
- The unitary model structure constructed in **[Dell'Ambrogio,2]**'s paper recognizes as weak equivalences *unitary equivalences of categories* (see **[Warner]**, ch. **17**, or better **[Dell'Ambrogio,2]**, Lemma **4.6**) mere equivalences of categories; cofibrations are *-functors injective on objects. Proposition **4.15** in **[Dell'Ambrogio,2]** shows that the (2-)category of C*-categories is cofibrantly generated by the sets

$$I = \{\varnothing \to \mathbb{C}, \mathbb{C} \amalg \mathbb{C} \to 1, P \to 1\}$$
$$J = \{\mathbb{C} \to 1\}$$

where $\varnothing$ is the empty category, 1 is the discrete category $\{0,1\}$ and P is the pushout of the diagram $1 \leftarrow \mathbb{C} \amalg \mathbb{C} \to 1$ (compare this result with a formally analogous statement for groupids, in **[Warner]**, ch. **15**).

- A *simplicial model category* is a model category which is enriched over the category **SSet** of simplicial sets, regarded as a closed model category with respect to Quillen's structure. In particular the enrichment is compatible with the monoidal structure on **SSet**, and renders C*-**Cat** tensored and cotensored.

The simplicial enrichment is deduced from the following tetrahedron of adjoint functors:



where the adjunction $\pi\colon \mathrm{C}^*\textbf{-Cat} \underset{\mathcal{G}_{\max}}{\overset{\mathrm{uni}}{\rightleftarrows}} \mathbf{Gpd} \underset{\Pi_1}{\overset{N}{\rightleftarrows}} \mathbf{SSet}\colon \nu$ is obtained via

$$\hom_{\mathrm{C}^*\textbf{-Cat}}(\mathcal{G}_{\max}(\Pi_1(A)),B) \cong \hom_{\mathbf{Gpd}}(\Pi_1 A, \mathrm{uni}\, B) \cong \hom_{\mathbf{SSet}}(A, N(\mathrm{uni}\, B))$$

(the adjunction **Gpd** $\leftrightarrows$ **SSet** is classical, C*-**Cat** $\leftrightarrows$ **Gpd** is constructed explicitly in **[Dell'Ambrogio,2]**, §**3.2**).

This adjunction is a Quillen pair, hence it defines a simplicial enrichment turning C*-**Cat** into a simplicial **SSet**-algebra in the sense of Hovey.

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