The algebra and geometry of categorical groups

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ma per seguir virtute e canoscenza

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Algebra

Sketches of a Group

Categorical groups come in various (equivalent) flavours:

- Groups internal to **Gpd** (a category with a bifunctor $\cdot\colon \mathbf{G}\times\mathbf{G}\to\mathbf{G}$ which turns it into a group in a way –the Eckmann-Hilton relation– which is compatible with composition: $(f\circ g)\cdot(h\circ k) = (f\cdot h)\circ(g\cdot k)$);
- Crossed modules (=categories internal to **Grp**, i.e. pairs of groups G_{ob}, G_{ar} with source-target maps $G_{ar}\rightrightarrows G_{ob}$, a composition $G_{ar}\times_{G_{ob}} G_{ar}\to G_{ar}\dots$);
- 2-groupoids with a single object: $\begin{tikzcd} f \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & g \end{tikzcd}$;
- ...

A crossed module consists of a pair of groups G, E, an action of G on E and a morphism $\partial\colon E\to G$ such that (Peiffer conditions)

- $\partial(g.e) = g\partial e g^{-1}$;
- $\partial e.f = efe^{-1}$.

There is an obvious notion of morphism of crossed modules.

All these different approaches are equivalent: for example

Theorem (Verdier)

There is an equivalence of categories between groups internal to **Gpd** and crossed modules.

See [Fo] for a detailed and modern proof.

Sketch of one side of the proof: given a group in **Gpd** we can define the crossed module having $G = Ob(\mathbf{G})$, $E = \coprod_{X \in Ob(\mathbf{G})} \mathbf{G}(1, X)$, $\partial = t$ (the target map): $(e \colon 1 \to X) \mapsto X$.

Definition (Notation)

We will denote $\mathbf{G}_{E,G}$ the categorical group associated to the crossed module (E, G, a, ∂), and $E_{\mathbf{G}}$, $G_{\mathbf{G}}$ the (two groups of the) crossed module associated to a categorical group $\mathbf{G}$.

Examples

- The delooping of an abelian[1] group G, i.e. the category $\mathbf{B}G$ having a single object $*$ and such that $\mathbf{B}G(*,*) = G$. Notice that for an abelian group K, $E(\mathbf{B}K) = K$, $G(\mathbf{B}K) = 1$.
- The categorical group associated to a category $\mathcal{C}$, having as objects the invertible endofunctors and as arrows the invertible natural isomorphisms between such functors. If $\mathcal{C} = \mathbf{B}K$, then the corresponding c.m. is $K \to \mathrm{Aut}(K) \colon g \mapsto g(-)g^{-1}$. There is a functor **Cat** $\to$ **CatGrp** which sends a category to its associated c.g.

Definition (Notation)

A c.g. can be delooped too[a]: given $\mathbf{G}$ its delooping is the 2-category $\mathbf{BG}$ with a single object $*$ and such that $\mathbf{BG}(*,*)$ is the category $\mathbf{G}$. Composition is defined as the group operation in $(\mathbf{G},\cdot)$ at the level of 1-cells, and as the composition in the category $\mathbf{G}$ at the level of 2-cells.

[a] It is a common procedure in monoidal category theory working in full generality.

[1] If the interchange law has to be true, we must consider only *abelian* groups.

A categorical group is said to be

- free if there is at most one arrow between any two distinct objects (so that $\partial\colon E(\mathbf{G}) \to G(\mathbf{G})$ is mono and E embeds in G as a normal subgroup);
- transitive if there is an arrow between any two objects (so that $\partial\colon E(\mathbf{G}) \to G(\mathbf{G})$ is epi).
- intransitive if there are no arrows between any two distinct objects (so that $\partial\colon E(\mathbf{G}) \to G(\mathbf{G})$ is the trivial map). Example: the wreath product $\mathbb{C}^\times \wr Sym(n)$.

Recall a couple of notions from 2- and 3-category theory: a 2, 3-category consists of objects (0-cells), arrows (1-cells) and

- 2-cells (Example: natural transformations in **Cat**);
- 3-cells (Example: modifications between natural transformations in the category of all 2-categories);
- composition of 2- and 3-cells.

A natural 2-transformation, between two 2-functors $F, G\colon \mathbf{C} \to \mathbf{D}$ consists of a map $h\colon Ob(\mathbf{C}) \to Mor(\mathbf{D})$ such that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{C}(X,Y) \arrow[r] \arrow[d] & \mathbf{D}(GX,GY) \arrow[d] \\
\mathbf{C}(FX,FY) \arrow[r] \arrow[ur, Rightarrow, "\cong", "\tilde{h}(X,Y)"'] & \mathbf{D}(FX,GY)
\end{tikzcd}$$

where $\tilde{h}(X,Y)$ is a natural iso $h(Y)\circ F(f) \Rightarrow G(f)\circ h(X)$ such that

- for any $f\colon X \to Y$, $g\colon Z \to X$, ($\tilde{h}(f) := \tilde{h}(X,Y)$, $\tilde{h}(g) := \tilde{h}(Z,X)$ for short), the following commutes:

$$\begin{tikzcd}
G(f)\circ h(X)\circ F(g) \arrow[r, equal] & G(f)\circ h(X)\circ F(g) & h(X) \arrow[dd, Rightarrow, "\tilde{h}(1_X)=1_{h(X)}"] \\
h(Y)\circ F(f)\circ F(g) \arrow[u, Rightarrow, "\tilde{h}(f)*F(g)"] & G(f)\circ G(g)\circ h(Z) \arrow[u, Rightarrow, "G(f)*h(g)"'] & \\
h(Y)\circ F(fg) \arrow[u, equal] \arrow[r, Rightarrow, "\tilde{h}(fg)"'] & G(fg)\circ h(Z) \arrow[u, equal] & h(X)
\end{tikzcd}$$

A modification $\Theta\colon h \Rrightarrow k$ between two natural 2-transformation $(h,\tilde{h}),(k,\tilde{k})\colon F \to G$, where both F, G are functors $\mathbf{C} \to \mathbf{D}$, consists of a function Θ assigning to any $X \in Ob(\mathbf{C})$ a 2-cell $\Theta_X\colon h(X) \Rightarrow k(X)$, in such a way that

$$\begin{tikzcd} G(f)\circ h(X) \arrow[r, Rightarrow, "G(f)*\Theta_A"] & G(f)\circ k(X) \\ h(Y)\circ F(f) \arrow[u, Rightarrow, "\tilde{h}(f)"] \arrow[r, Rightarrow, "\Theta_B * F(f)"'] & k(Y)\circ F(f) \arrow[u, Rightarrow, "\tilde{k}(f)"'] \end{tikzcd}$$

(hor) 2-cells can be composed: $\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow h"] & \mathbf{B} \arrow[r, bend left, "U"] \arrow[r, bend right, "V"'] \arrow[r, phantom, "\Downarrow k"] & \mathbf{C} \end{tikzcd}$ goes to $k \boxminus h$: $\begin{tikzcd} A \arrow[r, bend left, "U\circ F"] \arrow[r, bend right, "V\circ G"'] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd}$;

(ver) 2-cells can be composed: $\begin{tikzcd} \mathbf{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow h"] \arrow[r, phantom, shift right=3, "\Downarrow k"] & \mathbf{B} \end{tikzcd}$ goes to $k \boxbar h$: $\begin{tikzcd} \mathbf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow"] & \mathbf{B} \end{tikzcd}$;

- 3-cells $\Theta, \Psi\colon h \Rrightarrow k$ compose similarly horizontally and vertically.

The Kapranov-Voevodsky category 2-**Vect**

The category 2-**Vect** of KV 2-vector spaces has

- as objects the set $\mathbb{N}$ of (nonzero) natural numbers;
- as 1-cells $A\colon N \to M$, all $N \times M$ matrices with natural coefficients, $A = (a_{ij}) \in \mathbb{N}^{N\times M}$;
- as 2-cells $\begin{tikzcd} N \arrow[r, bend left, "A"{name=U}] \arrow[r, bend right, "B"'{name=D}] & B \arrow[from=U, to=D, phantom, "\Downarrow\Theta"] \end{tikzcd}$ all $N \times N$ matrices of matrices whose (i,j)-entry is an $a_{ij} \times b_{ij}$ matrix.

Example: $2, 3$ are objects of 2-**Vect**; a 1-morphism $2 \to 3$ consists of an integer valued 2×3 matrix, e.g. $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{smallmatrix}\right)$ or $B = \left(\begin{smallmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{smallmatrix}\right)$; a 2-morphism between A and B is, for example, a matrix like

$$\Theta = \begin{pmatrix} a & \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} & \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix} \\ \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} & \begin{pmatrix} e_1 & e_2 \end{pmatrix} & \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \end{pmatrix}$$

The category of 2-vector spaces has

- An horizontal composition between 2-cells,

$$\begin{tikzcd} N \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\Theta"] & M \arrow[r, bend left, "C"] \arrow[r, bend right, "D"'] \arrow[r, phantom, "\Downarrow\Xi"] & P \end{tikzcd} \rightsquigarrow \begin{tikzcd} N \arrow[rr, bend left, "CA"] \arrow[rr, bend right, "DB"'] \arrow[rr, phantom, "\Downarrow\Xi\boxminus\Theta"] & & P \end{tikzcd}$$

where CA, DB are obtained via product of integer matrices, and $\Xi \boxminus \Theta$ is the complex matrix whose (i,j)-entry has size $\sum a_{ik}c_{kj} \times \sum b_{ih}d_{hj}$, hence it is obtained as $\bigoplus_{k=1}^{M} \Xi_{ij} \otimes \Theta_{kj}$.

- A vertical composition

$$\begin{tikzcd} N \arrow[r, bend left=60, "A"] \arrow[r, "B" description] \arrow[r, bend right=60, "C"'] \arrow[r, phantom, bend left=30, "\Downarrow\Theta"] \arrow[r, phantom, bend right=30, "\Downarrow\Xi"] & M \end{tikzcd} \rightsquigarrow \begin{tikzcd} N \arrow[rr, bend left, "A"] \arrow[rr, bend right, "C"'] \arrow[rr, phantom, "\Downarrow\Xi\boxbar\Theta"] & & M \end{tikzcd}$$

, where $(\Xi \boxbar \Theta)_{ij} = \Xi_{ij}\Theta_{ij}$ (mult. of matrices). It makes sense, because Ξ_{ij} is a $a_{ij} \times b_{ij}$-matrix, and Θ_{ij} is a $b_{ij} \times c_{ij}$-matrix.

Verify the interchange law:

$$(\Omega \boxminus \Psi) \boxbar (\Xi \boxminus \Theta) = (\Xi \boxbar \Omega) \boxminus (\Psi \boxbar \Theta)$$

$$
\begin{tikzcd}
N \arrow[r, "A"] \arrow[d, equal] \arrow[dr, phantom, "\Theta"] & M \arrow[r, "C"] \arrow[d, equal] \arrow[dr, phantom, "\Xi"] & P \arrow[d, equal] \\
N \arrow[r, "B"'] \arrow[d, equal] \arrow[dr, phantom, "\Psi"] & M \arrow[r, "D"'] \arrow[d, equal] \arrow[dr, phantom, "\Omega"] & P \arrow[d, equal] \\
N \arrow[r, "E"'] & M \arrow[r, "F"'] & P
\end{tikzcd}
\longmapsto
\begin{tikzcd}
N \arrow[rr, "CA"] \arrow[d, equal] \arrow[drr, phantom, "\Xi \boxminus \Theta"] & & P \arrow[d, equal] \\
N \arrow[rr, "DB"'] \arrow[d, equal] \arrow[drr, phantom, "\Omega \boxminus \Psi"] & & P \arrow[d, equal] \\
N \arrow[rr, "FE"'] & & P
\end{tikzcd}
$$

$$
\downarrow \qquad\qquad\qquad\qquad \downarrow
$$

$$
\begin{tikzcd}
N \arrow[r, "A"] \arrow[d, equal] \arrow[dr, phantom, "\Psi \boxbar \Theta"] & M \arrow[r, "C"] \arrow[d, equal] \arrow[dr, phantom, "\Omega \boxbar \Xi"] & P \arrow[d, equal] \\
N \arrow[r, "E"'] & M \arrow[r, "F"'] & P
\end{tikzcd}
\longmapsto
\begin{tikzcd}
N \arrow[rr, "CA"] \arrow[d, equal] \arrow[drr, phantom, "{\begin{array}{c}(\Omega \boxminus \Psi) \boxbar (\Xi \boxminus \Theta), \\ (\Xi \boxbar \Omega) \boxminus (\Psi \boxbar \Theta)\end{array}}"] & & P \arrow[d, equal] \\
N \arrow[rr, "FE"'] & & P
\end{tikzcd}
$$

The category of 2-vector spaces has

- A monoidal product $\boxtimes$, such that
  - $N \boxtimes M$ is the product of natural numbers;
  - Given $A \colon N \to M$, $B \colon R \to S$, $A \boxtimes B \colon NR \to MS$, where $(A \boxtimes B)^{ij}_{kl} = a_{ik} b_{jl}$.
  - Given $\Theta, \Psi \colon A \to B$, $(\Theta \boxtimes \Psi)^{ij}_{kl} = \Theta_{ik} \otimes \Psi_{jl}$ (tensor product of matrices).
- A monoidal sum such that $N \boxplus M$ is the sum of natural numbers, $A \boxplus B$ is the direct sum of matrices, and $\Theta \boxplus \Psi$ is the componentwise direct sum of matrices.
- A direct sum in any 2-**Vect**(N, M), where $(A \oplus B)_{ij} = (a_{ij} + b_{ij})$, componentwise sum of integer matrices, and $(\Theta \oplus \Psi)_{ij} = \Theta_{ij} \oplus \Psi_{ij}$, direct sum of complex matrices.
- A sum, which defines the linear structure on each 2-**Vect**(N, M), since $(\Theta + \Psi)_{ij}$ is obtained summing the matrices Θ_{ij}, Ψ_{ij} (they have the same size $a_{ij} \times b_{ij}$).

Verify: maybe (2-**Vect**$, \boxtimes, \boxplus$) is a rig-category?

Categorical Representations

Definition

We define $\mathbf{GL}(N) \subset 2\text{-}\mathbf{Vect}(N,N)$ to be the categorical group associated to the category $2\text{-}\mathbf{Vect}(N,N)$. It is called the general 2-linear categorical group of dimension N.

Remark

The crossed module associated to $\mathbf{GL}(N)$ is the wreath product $\mathbb{C}^\times \wr Sym(N)$, where

- The map $\partial \colon (\mathbb{C}^\times)^N \to Sym(N)$ is the trivial one sending everything to id;
- the action of $Sym(N)$ on $(C^\times)^N$ permutes the coordinates of a vector.

A (linear) representation of a group G can be regarded as a functor from the delooping of G, i.e. $\rho \colon \mathbf{B}G \to \mathbf{Vect}$; hence we are led to define a linear 2-representation of a categorical group $\mathbf{G}$ as a 2-functor $R \colon \mathbf{BG} \to \mathbf{2\text{-}Vect}$.

The integer $R(*) = N$ is called the *dimension* of the representation.

The category $\mathrm{Rep}_{2\text{-}\mathbf{Vect}}(\mathbf{G}) = \mathrm{Fun}(\mathbf{BG}, 2\text{-}\mathbf{Vect})$ is a 2-category where 1-cells are called intertwiners between representations, and 2-cells are called 2-intertwiners: these are, respectively, natural transformations between representation $R, T \colon \mathbf{BG} \to 2\text{-}\mathbf{Vect}$, and modification between natural transformations $h, k \colon R \Rightarrow T$.

$\mathrm{Rep}_{2\text{-}\mathbf{Vect}}(\mathbf{G})$ becomes a monoidal category

- with respect to a monoidal product of representations, obtained componentwise as $(R, \tilde{R}), (T, \tilde{T}) \mapsto R \boxtimes T(X) = R(X) \boxtimes T(X)$, with a suitable coherer $\widetilde{R \boxtimes T}$.
- With respect to a monoidal sum of representations obtained componentwise as $(R, \tilde{R}), (T, \tilde{T}) \mapsto R \boxplus T(X) = R(X) \boxplus T(X)$ (the coherer of this monoidal structure is trivial).

Both monoidal structure extend by (2-)functoriality to 1- and 2-intertwiners, following the rules of the former definition for $\boxtimes$ and $\boxplus$.

Strict representations

We can now turn on the study of *strict* 2-representations of categorical groups: these are strict functors $\rho\colon \mathbf{BG} \to 2\text{-}\mathbf{Vect}$, which can be regarded as morphisms of c.g.

$$\bar{\rho}\colon \mathbf{G} \to \mathbf{GL}(\rho(*)) = \mathbf{GL}(N)$$

hence equivalently as morphisms of crossed modules
$$\begin{tikzcd} E_{\mathbf{G}} \arrow[r] \arrow[d, "\rho_{\boldsymbol{t}}"'] & G_{\mathbf{G}} \arrow[d, "\rho_{\boldsymbol{b}}"] \\ (\mathbb{C}^\times)^N \arrow[r, "1"'] & Sym(N) \end{tikzcd} .$$

1-intertwiners between strict representations ρ^N, σ^M can be described as arrows $h_* \in 2\text{-}\mathbf{Vect}(N, M)$ with $\tilde{h}\colon X \mapsto \tilde{h}(X) \in 2\text{-}\mathbf{Vect}(h_* \circ \rho(X), \sigma(X) \circ h_*)$, subject t certain coherence conditions:

$$\begin{tikzcd} h_*\rho(XY) \arrow[rr, Rightarrow, "\tilde{h}(XY)"] \arrow[d, equal] & & \sigma(XY)h_* \arrow[d, equal] \\ h_*\rho(X)\rho(Y) \arrow[r, Rightarrow, "\tilde{h}(X)\rho(Y)"'] & \sigma(X)h_*\rho(Y) \arrow[r, Rightarrow, "\sigma(X)\tilde{h}(Y)"'] & \sigma(X)\sigma(Y)h_* \end{tikzcd} \qquad \begin{tikzcd} h_*\rho(1) = h_* \arrow[d, equal, "1_{h_*}"] \\ \sigma(1)h_* = h_* \end{tikzcd}$$

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$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \rho(X)h_* \arrow[r, Rightarrow, "\tilde{h}(X)"] \arrow[d, Rightarrow, "\rho(f)*h_*"'] & h_*\sigma(X) \arrow[d, Rightarrow, "h_**\sigma(f)"] \\ \rho(Y)h_* \arrow[r, Rightarrow, "\tilde{h}(Y)"'] & h_*\sigma(Y) \end{tikzcd}$$

Example: $\mathbf{G}(3,2)$

Let (C_3, C_2, ∂, a) the crossed module where $C_2 = \{\pm 1\}$, $C_3 = \{1, x, x^{-1}\}$, $\partial\colon C_3 \to C_2$ is the trivial map and the action $a\colon C_2 \curvearrowright C_3$ coincides with the inversion $x \mapsto x^{-1}$. Let $\mathbf{G}(3,2)$ be the associated categorical group.

We can classify all the 1- and 2-dimensional 2-representations of $\mathbf{G}(3,2)$:

$\mathcal{V}(1)$ Trivial 1-dimensional representation:
$$\begin{tikzcd} C_3 \arrow[r, "\partial"] \arrow[d, "\rho_{\boldsymbol{t}}"'] & C_2 \arrow[d, "\rho_{\boldsymbol{b}}"] \\ \mathbb{C}^\times \arrow[r, "1"'] & 1 \end{tikzcd}$$

There are no other 1-dimensional representations, since ρ_t is determined by a character of C_3, which is forced to be the trivial one by the commutativity condition.

$\mathcal{V}(2)$ Trivial 2-dimensional representation:
$$\begin{tikzcd} C_3 \arrow[r, "\partial"] \arrow[d, "\rho_{\boldsymbol{t}}"'] & C_2 \arrow[d, "\rho_{\boldsymbol{b}}"] \\ (\mathbb{C}^\times)^2 \arrow[r, "1"'] & C_2 \end{tikzcd} \; : \; \mathcal{V}(2) \cong \mathcal{V}(1) \boxplus \mathcal{V}(1).$$

All the other 2-dimensional representations of $\mathbf{G}(3,2)$ are classified by the ring of characters of C_3:

$$\begin{cases} \rho_b(\pm 1) = \pm 1 \in C_2 \\ \rho_t(x) = (\xi(x), \psi(x)) \end{cases}$$

where ξ, ψ are complex characters of C_3: but now $\xi = \psi^{-1}$, by the commutativity relation for (ρ_t, ρ_b) to be a morphism of crossed modules. So ρ is completely determined by ξ.

One can also classify all the 1- and 2-intertwiners between these representations: it is done in [Mac] pp. 21-25.
Hao about the structure of monoidal bicategory of $\mathrm{Rep}(\mathbf{G}(2,3))$? As an example: the trivial 1-dimensional character/representation acts as a neutral element for $\boxtimes$.

Characters

A character of a categorical grup $\mathbf{G}$ is a 1-dimensional 2-representation $\bar{\rho}\colon \mathbf{G} \to \mathbf{GL}(1)$.

These representations are completely classified by a complex character ξ_ρ of the group $E_{\mathbf{G}}$ of the associated crossed module, $E_{\mathbf{G}} \rightsquigarrow \mathbb{C}^\times$. An intertwiner $(h, \tilde{h})$ between two such characters is either zero or it is completely determined by a representation $\tilde{h}$ of the group $G_{\mathbf{G}}$, acting as

$$\tilde{h}(X) = \xi)\rho(e)^{-1}\xi_\sigma(e)$$

where e is such that $\partial e = X \in G_{\mathbf{G}}$.
A 2-intertwiner $\Theta\colon (h, \tilde{h}) \to (k, \tilde{k})$ consists simply of a classical intertwiner between the two representation $\tilde{h}$ and $\tilde{k}$.

Geometry

Classical Čech theory

Suppose M is a manifold, and G a topological group;

- For any covering $\{U_i\}_{i\in I} = \mathfrak{U}$ of M we can organize isomorphism classes of principal G-bundles in a set $\check{H}^1(\mathfrak{U}, G)$,
- which is in bijection with

$$\left\{ {\text{1-cocycles} \atop g_{ij}: U_i \cap U_j \to G} \right\} \Big/ {\text{being} \atop \text{cohomologus}}$$

  where "being cohomologous" for two cocycles g_{ij}, g'_{ij} (i.e. functions satisfying the *cocycle condition* for any $i, j, k \in I$) means that there is a family of functions $\{f_i\}$ such that $g_{ij} f_j = f_i g'_{ij}$.
- If M can be "nicely" covered, then $\check{H}^1(\mathfrak{U}, G) = \check{H}^1(M, G)$ does not depend on the choice of a good cover $\mathfrak{U}$; otherwise one obtains $\check{H}^1(M, G)$ as a limit $\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, G)$ over all covers $\mathfrak{U} \in \mathrm{cov}(M)$ (notice that good covers are cofinal in this family).
- (Brown) The functor $\check{H}^1(-, G)$ is representable in the homotopy category of spaces, and its representing objects is the classifying space of G: $\check{H}^1(M, G) \cong [M, \mathrm{B}G]$.

Can one obtain a similar theory for (whatever they are) principale G-2-bundles for a (topological) categorical group G?

First of all the basic theory extends *verbatim* adding everywhere the word "topological":

- A topological categorical group is an internal group in **TopGpd**, the category of toplogical groupoids;
- A topological crossed module consists of an internal category in **TopGrp**, the category of topological groups;
- Verdier's theorem applies as well: (topological) crossed modules are equivalent to (topological) categorical groups.
- However a new equivalent characterization of a c.g. can be used to better understand the topological behaviour of such things...

(Topological) categorical groups are (topological) 2-groupoids with a single object.

Let $\mathbf{G}$ be a c.g., $(E_{\mathbf{G}}, G_{\mathbf{G}})$ be its associated crossed module. As we have seen before, $\mathbf{G}$ can be delooped to a 2-category with a single object $*$, where

- 1-cells are objects of $\mathbf{G}$ and an arrow $g \to g'$ is $h \in E$ such that $\partial(h)g = g'$.
- Composition of 1-cells amounts to the monoidal product, and inversion equals dualization,
- horizontal and vertical composition are obtained thanks to Peiffer relations between $\partial \colon E \to G$ and $a \colon G \curvearrowright E$, and the interchange law holds because ∂ and a are suitably compatible.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "\partial(h)g"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "\partial(k)f"'] \arrow[r, phantom, "\Downarrow k"] & \bullet \end{tikzcd} \rightsquigarrow \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, bend left=35, "\Downarrow"] \arrow[r, phantom, bend right=35, "\Downarrow"] & \bullet \end{tikzcd} \rightsquigarrow \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

This category is easily seen to be a groupoid!

Segal fundamental Lemma

Čech cocycles are functors!

Let $\{U_i\} = \mathfrak{U}$ be a covering of the base space M; then we can define a groupoid $\widehat{\mathfrak{U}}$ having objects $\coprod_{i\in I} U_i$, and a unique arrow $(x,i) \to (x,j)$ iff $x \in U_i \cap U_j$; define a functor $\hat{g}\colon \widehat{\mathfrak{U}} \to G\colon (x,i) \mapsto *$ (the unique object of G regarded as a category), and sending an arrow $x \in U_{ij}$ to an element g_{ij} of G;

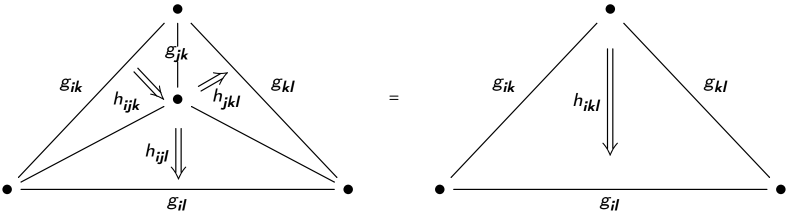
$$\left\{ \begin{matrix} \text{cocycles} \\ \{g_{ij}\} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{functors} \\ \hat{g}\colon\widehat{\mathfrak{U}}\to G \end{matrix} \right\}$$

- Functoriality conditions for $\hat{g}$ are precisely cocycle conditions for $\{g_{ij}\}$;
- two cocycles are cohomologous iff their corresponding functors are naturally isomorphic.

We can categorify Segal's fundamental Lemma:

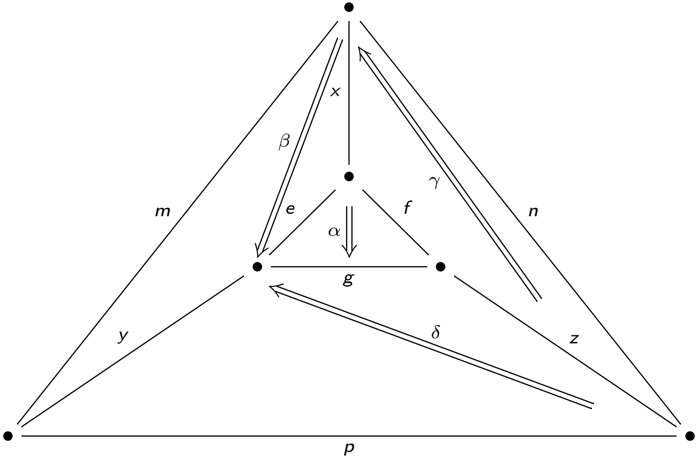
- A Čech cocycle for a categorical group consists in this setting of a pseudofunctor $\hat{g}\colon \widehat{\mathfrak{U}} \to \mathbf{G}$, where $\widehat{\mathfrak{U}}$ is the groupoid of the Segal's construction regarded as a 2-category having only identity 2-cells, and $\mathbf{G}$ a 2-groupoid with a single object $\bullet$.

Pseudofunctoriality now translates into some coherence conditions having these shapes: a suitable tetrahedron for the cocycle cond.,



$$\begin{tikzcd} {} \arrow[r, Rightarrow, "g_{jl}\cdot h_{ijk}"] \arrow[d, Rightarrow, "h_{jkl}\cdot g_{ik}"'] & {} \arrow[d, Rightarrow, "h_{ijl}"] \\ {} \arrow[r, Rightarrow, "h_{ikl}"'] & {} \end{tikzcd}$$

and the prism



which expresses the compatibility between cohomologus cocycles/isomorphic functors $\widehat{\mathfrak{U}} \to \mathbf{G}$, and is equivalent to

$$\begin{tikzcd}
 & \bullet \arrow[dl, dash, "m"'] \arrow[dr, dash, "n"] \arrow[d, Rightarrow, "\zeta"'] & \\
\bullet \arrow[rr, dash, "p"'] & {} & \bullet
\end{tikzcd}.$$

with the additional coherence condition

$$\begin{tikzcd}
znm \arrow[r, Rightarrow, "\gamma\cdot m"] \arrow[d, Rightarrow, "z\cdot\zeta"'] & fxm \arrow[r, Rightarrow, "f\cdot\beta"] & fey \arrow[d, Rightarrow, "\alpha\cdot y"] \\
zp \arrow[rr, Rightarrow, "\delta"'] & & gy
\end{tikzcd}$$

These conditions are exactly the coherence conditions which in [Ba] define principal **G**-2-bundles over a manifold M:

- Let (E, G, ∂, a) be the crossed module associated to **G**, and $\{U_i\} = \mathcal{U}$ a cover of M.
- Then a cocycle subordinated to $\mathcal{U}$ consists of maps $g_{ij}\colon U_{ij} \to G$ such that
  - A weak cocycle condition is satisfied:

$$\partial(h_{ijk}) g_{ij} g_{jk} = g_{ik}$$

    for some $h_{ijk}\colon U_i \cap U_j \cap U_k \to H$.
  - Similarly, we say two weak cocycles (g_{ij}, h_{ijl}) and (g'_{ij}, h'_{ijl}) are cohomologous if there is a family of maps $f_i\colon U_i \to G$, $k_{ij}\colon U_i \cap U_j \to H$ such that

$$\partial(k_{ij}) g_{ij} f_j = f_i g'_{ij}.$$

"Being cohomologous" is now an equivalence relation on the set $\check{Z}(\mathcal{U}, \mathbf{G})$ of cocycles subordinated to $\mathcal{U}$, for the categorical group $\mathbf{G}$; we define $\check{H}^1(\mathcal{U}, \mathbf{G})$ to be the quotient of $\check{Z}(\mathcal{U}, \mathbf{G})$ by this relation, and

$$\check{H}^1(M, \mathbf{G}) := \varinjlim_{\mathcal{U} \in \mathrm{cov}(M)} \check{H}^1(\mathcal{U}, \mathbf{G}).$$

Main Theorem: Let $\mathbf{G}$ a well pointed[2] topological c.g. and M a manifold. Then

$$\check{H}^1(M, \mathbf{G}) \cong [M, \mathrm{B}|\mathbf{G}|]$$

where $\mathrm{B}|\mathbf{G}|$ is the classifying space of the topological group $|\mathbf{G}|$, defined as the geometric realization of the nerve of the category $\mathbf{G}$.

[2]This is a technical condition for a topological c.g. in which the groups $E_{\mathbf{G}}, G_{\mathbf{G}}$ have the homotopy type of a cw-complex. A topological group, or a pointed space, is *well-pointed* if $\{1_G\} \to G$ is a closed cofibration; a c.g. is well pointed, by definition, if $E_{\mathbf{G}}, G_{\mathbf{G}}$ are well pointed.

The nerve-realization paradigm

Theorem (General nonsense adjunction)

Given a functor $F\colon \mathbf{C} \to \mathbf{D}$ to a cocomplete category, it is always possible to exhibit an extension of F to the category $\widehat{\mathbf{C}}$ of presheaves of sets on $\mathbf{C}$, $\overline{F} = \mathrm{Lan}_y F\colon \widehat{\mathbf{C}} \to \mathbf{D}$ (this is sort of a universal property of $\widehat{\mathbf{C}}$). Moreover this functor has a right adjoint S.

Apply the previous theorem to the following two cases:

$$
\begin{tikzcd}
\Delta \arrow[r, tail, "i"] \arrow[d, tail, "y"'] & \mathbf{Cat} \arrow[dl, shift left, "N"] \\
\widehat{\Delta} \arrow[ur, shift left] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta \arrow[r, "\rho"] \arrow[d, tail, "y"'] & \mathbf{Top} \arrow[dl, shift left] \\
\widehat{\Delta} \arrow[ur, shift left, "R"] &
\end{tikzcd}
$$

- $\triangleright$ $i\colon \Delta \to \mathbf{Cat}$ regards the totally ordered set $\{0 < 1 < \cdots < n\}$ as a category;
- $\triangleright$ y is the Yoneda embedding $\Delta \hookrightarrow \widehat{\Delta}\colon [n] \mapsto \Delta(-, [n])$;
- $\triangleright$ $\rho\colon \Delta \to \mathbf{Top}$ "realizes" each $\{0 < 1 < \cdots < n\}$ as the standard n-simplex embedded in $\mathbb{R}^{n+1}$.

$$
\begin{array}{ccccc}
\mathbf{Cat} & \longrightarrow & \widehat{\Delta} & \longrightarrow & \mathbf{Top} \\
\mathbf{G} & \longmapsto & N(\mathbf{G}) & \longmapsto & R(N(\mathbf{G})) = |\mathbf{G}|
\end{array}
$$

The proof of the Main Theorem exploits the following three lemma:

1. Let $\mathbf{G}$ a topological well pointed c.g.; then
   - $|\mathbf{G}|$ is (a topological group and) again well pointed;
   - There exist a c.g. $\tilde{\mathbf{G}}$ and a short exact sequence of topological groups $1 \to H \to |\tilde{\mathbf{G}}| \to |\mathbf{G}| \to 1$ where H is the domain of ∂ in the crossed module associated to $\mathbf{G}$;
   - $\tilde{\mathbf{G}} \cong G \ltimes EH$, where EH is the universal principal H-bundle over BH (corresponds to 1_{BH} in $[BH, BH] \cong \check{H}^1(BH, H)$).
2. Any short exact sequence $\sigma \colon 1 \to H \xrightarrow{\trianglelefteq} G \to K \to 1$ gives a crossed module $\mathbf{G}_\sigma = (G, H)$ where the action of G is by conjugation and ∂ is the inclusion $H \hookrightarrow G$; then there is an isomorphism

$$\check{H}^1(M, \mathbf{G}(G, H)) := \check{H}^1(M, \mathbf{G}_\sigma) \cong \check{H}^1(M, K)$$

3. Let $1 \to \mathbf{G} \to \mathbf{H} \to \mathbf{K} \to 1$ be a s.e.s. of categorical groups (it means that the underlying s.e.s. of topological groups obtained via associated crossed modules are exact): then there is a short exact sequence of pointed sets

$$\check{H}^1(M, \mathbf{G}) \to \check{H}^1(M, \mathbf{H}) \to \check{H}^1(M, \mathbf{K})$$

We know that there exists an isomorphims in classical Čech theory

$$\check{H}^1(M, |\mathbf{G}|) \cong [M, \mathrm{B}|\mathbf{G}|]$$

hence to prove the Theorem it suffices to show that $\check{H}^1(M, \mathbf{G}) \cong \check{H}^1(M, |\mathbf{G}|)$.
Now **Lemma 1** entails that there exists a short exact sequence σ

$$1 \to H \to G \ltimes EH \to |\mathbf{G}| \to 1$$

and **Lemma 2** entails that $\check{H}^1(M, \mathbf{G}_\sigma) \cong \check{H}^1(M, |\mathbf{G}|)$. So to build the desired isomorphism it suffices to show that

$$\check{H}^1(M, \mathbf{G}_\sigma) \cong \check{H}^1(M, \mathbf{G})$$

Now consider the short exact sequence of crossed modules

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 1 \\
(\bigcirc)1 \arrow[r] & EH \arrow[r] & G \ltimes EH \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Lemma 3 entails that $\check{H}^1(M, EH) \to \check{H}^1(M, \mathbf{G}_\sigma) \to \check{H}^1(M, \mathbf{G})$ is exact. But $\check{H}^1(M, EH)$ is zero, because EH is contractible, and the sequence $\bigcirc$ is split exact. $\square$

- John C. Baez, Danny Stevenson, *The Classifying Space of a Topological 2-Group*, http://arxiv.org/abs/0801.3843.
- Toby Bartels, *Higher gauge theory I: 2-Bundles*, http://arxiv.org/abs/math/0410328.
- John W. Barrett, Marco Mackaay, *Categorical representations of categorical groups*, http://arxiv.org/abs/math/0407463.
- . Forrester-Barker, *Group objects and internal categories*, http://arXiv.org/abs/math/0212065v1.