Homotopical interpretation of stacks
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Studio Manassé - Ellentétes érzelmek, 1950

Contents

1 A short intro to Topos Theory.

> I have to state my position on the most controversial question in the whole of topos theory: how to spell the plural of *tòpos*. The reader will already have observed that I use the English plural; I do so because (in its mathematical sense) the word is not a direct derivative of its Greek root $\tau\acute{o}\pi o\varsigma$ but a back-formation from topology. I have nothing further to say on the matter, except to ask those *toposophers* who persist in talking about $\tau\acute{o}\pi o\iota$ whether, when they go out for a ramble on a cold day, they carry supplies of hot tea with them in $\theta\acute{\epsilon}\rho\mu\grave{o}\iota$.
>
> **[Johnstone]**, pag. XX

NOTATIONS AND CONVENTIONS. Sets are supposed to be all small with respect to a given Grothendieck universe $\mathcal{U}$; categories are all $\mathcal{U}$-small and denoted by boldface letters as $\mathbf{C}, \mathbf{D}$. Functors between them are denoted plainly as F, G, H or sometimes in lowercase letters f, g, h. Natural transformations are usually denoted by greek letters $\alpha\colon F \Rightarrow G$. The Yoneda embedding $\mathbf{C} \hookrightarrow [\mathbf{C}^{\text{op}}, \mathbf{Sets}] = \widehat{\mathbf{C}}$ is denoted as $\text{Y}\colon C \mapsto \text{Y}(C) = \mathbf{C}(-, C)$. Initial/terminal objects in a category are denoted respectively as $\emptyset, 0$ and $*, 1$ or suchlike symbols. The category of functors between $\mathbf{C}$ and $\mathbf{D}$ is denoted $[\mathbf{C}, \mathbf{D}]$.

INTRODUCTION. One of the most famous books about topos theory is the (still incomplete) monography *Sketches of an Elephant* (**[Johnstone2]**); the title was inspired to Johnstone by the Jain parable of the six (or four) blind men that coming across an elephant try to understand which kind of animal it is.

The first blind man, touching the trunk of the elephant, claimed the animal was like a sort of drain pipe. For another one whose hand reached its ear, the animal seemed like a kind of fan. As for another person, who touched its leg, he said: "I perceive the shape of the elephant to be like a pillar, or a tree". And the one who placed his hand upon its back said: "Indeed, the elephant is like a throne". Each of them presented a true aspect when he related what he had gained from experiencing the elephant. None of them had strayed from the true description of the elephant, yet they fell short of fathoming the true appearance of the elephant.

So are toposes: chimeric entities which can be viewed at the same time as generalized topological spaces, generalized *universes* where we can develop set theory in a constructive way, categorified versions of the notion of Heyting algebra, the ideal setting in which to develop the analytical aspects of nonstandard analysis, ...

The parable of the six blind men is intended to teach the Jain principle of *anekānta* "manifoldness of thought", pluralism and multiplicity of viewpoints), according to which truth and reality are differently perceived from diverse points of view, and no single point of view embodies a global truth. Similarly, there's no hope to capture the essence of topos theory without accepting to look at mathematics as a whole subject, and maybe the extent of a lifetime is not enough to get acquainted even with the surface of the topic. This said, the aim of this first introductory section is twofold:

i. Provide the reader with the minimal amount of intuition of the intrinsic *geometric* nature behind the notion of (Grothendieck) topos;

ii. Collect the minimal amount of theory needed to appreciate the central part of this note, devoted to link the geometric notion of *stack* used in algebraic geometry to the (fairly more homotopical) notion of *fibrant object* in a model category.

As for the first task, the notion of *Grothendieck topos* was a pleasant byproduct of Grothendieck and Deligne's battle against Weil conjectures during the '60s, and can be viewed as a "generalized topological space". The original idea of Grothendieck was to interchange the rôles of the two notions, seeing the former (topological spaces, or better *sheaves* on topological spaces) as a really particular case of the latter (toposes: sheaves on a *site*, i.e. -roughly speaking- on a category endowed with a notion of *covering* for any object U).

Such a general viewpoint can now be justified in several ways, and even if the most evident "killer application" is still the proof of Weil conjectures given in **[SGA4]**, there are plenty of applications to other fields of mathematics, or to the task of *unify* mathematics under the aegis of a single ubiquitous idea: here are the most important two insights according to Grothendieck's philosophy.

- Spaces are better described via *sheaves*, rather than via the collection of their open sets (which can be recovered from the category $\mathrm{Sh}(X)$, under really mild hypoteses on the spaces). More precisely, any space X can be "probed" through various kind of sheaves allowing to capture its different "facets", which can be merely topological (i.e. encoded in the sheaf of continuous functions $U \mapsto C^0(U)$), smooth (i.e. encoded in the sheaves of smooth functions $U \mapsto C^\kappa(U)$, where $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$), *complex*-smooth (i.e. encoded in the sheaf of holomorphic functions $U \mapsto H(U)$), or algebraic (i.e. encoded in the sheaf of *polynomial* functions $U \mapsto \mathcal{O}_X(U)$).

  Such a general point of view in Geometry can be traced back to Gel'fand-Naimark's theorem, asserting an (anti-)equivalence of categories

  $$\mathrm{C}^*\textbf{-Alg}_c \cong \textbf{LCHaus} \tag{1}$$

  between tha category of (locally compact Hausdorff) topological spaces and the category of (commutative) C^*-algebras; the topology of X is entirely recovered by the *spectrum* of its algebra of continuous global functions $C(X) = \{f : X \to \mathbb{C}\}$.

- The definition of (Grothendieck) topos is modeled in such a way that the following leading principle holds: the category $\mathrm{Sh}(X)$ of sheaves of sets on a topological space is the archetypal example of a topos; $\mathrm{Sh}(X)$ "mimics" in a suitable sense any geometric feature of X, because given suitable (and rather natural from a categorical wiepoint) definitions of "homotopy and (co)homology groups" of a topos, then[1] we have an isomorphism between those groups and the "classical" homotopy/homology

[1] At least in the case where X is sufficiently tame (for example when its homotopy type is that of a CW-complex): see the introduction to [**Moerdijk2**] for more informations. The case of cohomology is somewhat subtler and has been studied in full generality by [**Duskin**]; the technical property of *having a natural number object* is alwawys fulfilled at least by Grothendieck toposes (see [**Moerdijk**], ch. **VI**).

groups of X:

$$\pi_n(\mathrm{Sh}(X), p_0) \cong \pi_n(X, x_0)$$
$$H^n(\mathrm{Sh}(X), \mathbb{Z}) \cong H^n(X, \mathbb{Z})$$

1.1 Grothendieck toposes.

Often in mathematics one has to consider correspondences which, albeit exhibiting a "sheafy" behaviour, can't be reduced to mere sheaves on spaces, either because of size issues (the domain category might be far from being as small as the posetal category of open subsets of some topological space X) or because sheaf-conditions do not seem to apply in any reasonable sense.

So, at least at first sight, there's no hope to consider a category $\mathbf{C}$ and a functor $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ (i.e., a presheaf), and turn the question "is F a *sheaf*?" into a meaningful one. In a few words, the problem reduces to state the sheaf conditions

- For any $U \subseteq X$, and for any covering $\{U_i\}$ of U, if $s|_{U_i} = t|_{U_i}$ n $F(U_i)$ for any $i \in I$ the $s = t$ as sections in $F(U)$.
- For any $U \subseteq X$, and for any covering $\{U_i\}$ of U, if a family of sections $\{s_i \in F(U_i)\}$ is such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a section $s \in F(U)$ such that $s_i = s|_{U_i}$.

in a way which is purely categorical on the one hand, and geometrically meaningful on the other. It is a well-known truism that the two condition summarize to the exactness of the sequence

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \cap U_j) \tag{2}$$

(meaning that $F(U) \cong \mathrm{eq}(a,b)$, where the maps a, b are obtained via the UMP of the product). But now, what does "being covered" by a family $\{U_i\}$ mean, for an object $U \in \mathbf{C}$?

Grothendieck's intuition that sheaf axioms are all about "what is being covered" led to the following definition of a *site* or *category with a Grothendieck topology*:

Definition 1.1. Let $\mathbf{C}$ be a category with pullbacks. A *(Grothendieck) topology* on $\mathbf{C}$ consists of a function which assigns to any object U in $\mathbf{C}$ a collection $\mathrm{COV}_{\mathbf{C}}(U)$ of families of arrows[2] $\{f_i\colon U_i \to U\}_{i\in I}$, called *coverings* of U, such that

i. If $V \to U$ is an isomorphism, the singleton $\{V \to U\}$ is in $\mathrm{COV}_{\mathbf{C}}(U)$: in particular $\{1_U\colon U \to U\}$ is always a covering of U, for any $U \in \mathrm{Ob}(\mathbf{C})$ (this reads as "any open set $U \subseteq X$ is trivially covered by $\{U\}$");

ii. (change of base) If $\{U_i \to U\} \in \mathrm{COV}_{\mathbf{C}}(U)$ and $V \to U$ is any arrow in $\mathbf{C}$, then $\{U_i \times_U V \to V\} \in \mathrm{COV}_{\mathbf{C}}(V)$;

[2] Notice that you are considering a collection *each element of which* is a set-indexed family of arrows to U: this issue is in principle able to lead to serious set-theoretic hardships in absence of Grothendieck's axiom of universes.

iii. (refinement) If $\{f_i\colon U_i \to U\} \in \mathrm{COV}_{\mathbf{C}}(U)$ and for any $i \in I$ there is $\{g_{ij}\colon V_{ij} \to U_i\}_{j\in J(i)} \in \mathrm{COV}_{\mathbf{C}}(U_i)$, then $\{f_i \circ g_{ij}\colon V_{ij} \to U_i \to U\} \in \mathrm{COV}_{\mathbf{C}}(U)$.

A category $\mathbf{C}$ endowed with a (Grothendieck) topology $\mathrm{COV}_{\mathbf{C}}$ is called a *(Grothendieck) site* and it is denoted $(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$.

Notice that the combined action of base-change and refining axioms entails that if $\{U_i \to U\}, \{V_j \to U\} \in \mathrm{COV}_{\mathbf{C}}(U)$, then $\{U_i \times_U V_j \to U\} \in \mathrm{COV}_{\mathbf{C}}(U)$.

Example 1.1. Let X be a space, and denote with $\mathrm{Ouv}(X)$ the category of its open subsets, regarded as a poset and hence as a category. It should be a truism that coverings of objects in $\mathrm{Ouv}(X)$ are exactly coverings of open sets. We can indeed turn $\mathrm{Ouv}(X)$ into a Grothendieck site choosing $\mathrm{COV}_{\mathrm{Ouv}(X)}(U)$ precisely as the set of all coverings of U.

Axioms are easily checked once we noticed that if $V_1 \to U, V_2 \to U$ are arrows in $\mathrm{Ouv}(X)$ the $V_1, V_2 \subseteq U$. Then the pullback $V_1 \times_U V_2$ is the intersection $V_1 \cap V_2$.

Definition 1.2. Call a family of functions $\{f_i\colon U_i \to U\}_{i\in I}$ *jointly surjective* if $\bigcup_i f_i(U_i) = U$.

Example 1.2 (The classical site of topological spaces)**.** Consider the category **Top** of topological spaces and continuous maps and define a covering $\mathrm{COV}_{\mathrm{cl}}(U)$ of U as a jointly surjective family of continuous open embeddings $U_i \to U$. Notice that mere inclusions are not enough to define a site structure: axiom (i) in Definition **1.1** entails that any homeomorphism $V \cong U$ has to be considered as a covering of U.

Example 1.3 (The global étale site of topological spaces)**.** Consider again the category **Top** of topological spaces. For any space Y define a covering $\mathrm{COV}_{\text{ét}}(U)$ of Y as a jointly surjective family of local homeomorphisms $E \to Y$.

$(\mathbf{Top}, \mathrm{COV}_{\text{ét}})$ is the *global étale* site.

Example 1.4 (The small étale site of a scheme)**.** Let X be a scheme. Consider the full subcategory $(\mathbf{Sch}/X)_{\text{ét}}$ of $\mathbf{Sch}/X$, consisting of morphisms $U \to X$ locally of finite presentation and étale. If $U \to X$ and $V \to X$ are objects of $(\mathbf{Sch}/X)_{\text{ét}}$, then an arrow $U \to V$ over X is necessarily étale. A covering of $U \to X$ in the *small étale topology* consists of a jointly surjective collection of morphisms $U_i \to U$.

Let's continue to consider the category $\mathbf{Sch}/X$ of schemes over a fixed scheme X. There are a number of topologies that one can put on it. Here are the most useful.

Example 1.5 (The global étale topology)**.** A covering $\{f\colon U_i \to U\} \in \mathrm{COV}_{\text{ét}}(U)$ is a jointly surjective collection of étale maps locally of finite presentation.

Example 1.6 (The global Zariski site)**.** Here a covering $\{f\colon U_i \to U\} \in \mathrm{COV}_{\mathrm{Zar}}(U)$ is a collection of open embeddings covering (in the topological sense) U. As in the example of the global classical topology, an open embedding *must* be defined as a morphism $V \to U$ that gives an isomorphism of V with an open subscheme of U, and not simply as the embedding of an open subscheme.

Example 1.7 (The fppf site)**.** A covering $\{f\colon U_i \to U\} \in \mathrm{COV}_{\mathrm{fppf}}(U)$ is a jointly surjective collection of flat maps locally of finite presentation.

The abbreviation *fppf* stands for “fidèlement plat et de présentation finie”, which translates to *faithfully flat and finitely presented*.

Separation and gluing axioms, defining the sheaf condition, can be rephrased in any Grothendieck site $(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$, providing the notion of intersection is changed with that of *pullback* of the domains of elements of a covering:

Definition 1.3. Let $(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$ be a Grothendieck site and $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$ a presheaf. Then F is said to be

- A *separated* presheaf if, for any $\{f_i\colon U_i \to U\}_{i\in I} \in \mathrm{COV}_{\mathbf{C}}(U)$and $a, b \in F(U)$ such that $F(f_i)(a) = F(f_i)(b)$ for any $i \in I$, then $a = b$.
- A *sheaf* if the following condition is satisfied:

  Given a covering $\{f_i\colon U_i \to U\}_{i\in I}$ and a family of elements $\{a_i \in F(U_i)\}_{i\in I}$, if for any $i, j \in I$ we denote $p_i\colon U_i \times_U U_j \to U_i$, $p_j\colon U_i \times_U U_j \to U_j$, and assume that $F(p_i)(a_i) = F(p_j)(a_j)$, then there exists a unique $a \in F(U)$ such that $F(f_i)(a) = a_i$ for any $i \in I$.

The collection $\mathrm{Sh}(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$ happens to be a full reflective[3] subcategory of $\mathrm{PSh}(\mathbf{C}) = [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$; this means that a morphism $F \to G$ between two sheaves is nothing more than a natural transformation of functors.

Definition 1.4. A *Grothendieck topos* is defined to be a category $\mathcal{E}$ equivalent to $\mathrm{Sh}(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$, for some category $\mathbf{C}$ and Grothendieck topology $\mathrm{COV}_{\mathbf{C}}$ on $\mathbf{C}$.

1.1.1 Giraud characterization of toposes.

Giraud offered an intrinsic characterization of Grothendieck toposes as categories fulfilling suitable exactness conditions:

Theorem 1.1 (Giraud’s Theorem)**.** A category $\mathcal{E}$ is equivalent to a Grothendieck site $\mathrm{Sh}(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$ if and only if

- $\mathcal{E}$ is complete and cocomplete;
- $\mathcal{E}$ has *disjoint* and *pullback-stable coproducts*, i.e. for any family $\{E_i\}$ of objects of $\mathcal{E}$ the square

$$\begin{array}{ccc} \amalg_{i\in I} E_i & \longleftarrow & E_k \\ \uparrow & & \uparrow \\ E_k & \longleftarrow & \emptyset \end{array} \tag{3}$$

  is a pullback for any $j, k \in I$ ($\emptyset$ is the initial object of $\mathcal{E}$) and the functor $B \times_A -$ respects coproducts, for any $B \to A$ and family of arrows $E_i \to A$.

[3]In order to find an adjoint to the inclusione, try to mimic the sheafification functor $(-)^a\colon [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \to \mathrm{Sh}(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$. Morally, one has to identify section “agreeing on any covering of any object” and lift families of sections which are compatible on a covering of U to a section on the whole U.

- Given an equivalence relation $\rho\colon R \hookrightarrow E \times E$ on an object $E \in \mathcal{E}$, the coequalizer

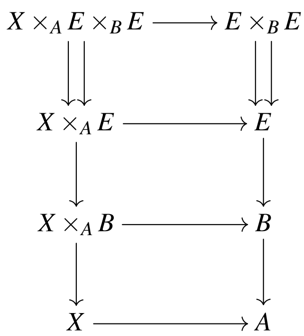
$$R \overset{\pi_1\rho}{\underset{\pi_2\rho}{\rightrightarrows}} E \longrightarrow C \tag{4}$$

exists (and is denoted E/R).

- Any epimorphism $p\colon E \twoheadrightarrow B$ admits a kernel pair (i.e. the pullback $E \overset{q_2}{\longleftarrow} E \times_B E \overset{q_1}{\longrightarrow} E$ exists) and p is the coequalizer of its kernel pair (i.e. the diagram

$$E \times_B E \overset{q_1}{\underset{q_2}{\rightrightarrows}} E \overset{p}{\longrightarrow} B \tag{5}$$

is exact).



- Given any diagram shaped like the one besides, for $X \to A$ any arrow in $\mathcal{E}$, the diagram

$$X \times_A E \times_B E \rightrightarrows X \times_A E \longrightarrow X \times_A B \tag{6}$$

is again exact.

- $\mathcal{E}$ admits a *generating family*, i.e. a set of objects $\{S_j\}$ such that $\{\mathcal{E}(S_j, -)\}$ is a jointly faithful family of functors.

1.2 Elementary toposes.

Fortunately it is not necessary to deal with such a nasty definition, because a more general and elegant notion, unraveled by Lawvere and Tierney, is available to us.

This shed some light on another interpretation of the notion of topos, namely that a topos is a "generalized universe of sets" (something even Grothendieck was aware of, but that he wasn't able to communicate in the right manner to logicians). In a few words, any topos can be regarded as offering an alternative model for ZF set theory (often without the Axiom of Choice, but if you do Geometry you probably want to get rid of it in any case), because any topos offers a setting where set theory can be rephrased in a suitable "internal semantics".

This note being intended to present the *geometric* side of the story, we will not deepen this (rather vast) topic, but we address the interested reader to **[Moerdijk]**, chap. **VI**, and to the whole work of **[Johnstone]**.

The main insight in Lawvere-Tierney definition of an "elementary" topos is that of a *subobject* (or *truth*) *classifier*:

Definition 1.5. An category $\mathcal{E}$ is an *elementary topos* if

- $\mathcal{E}$ is *finitely* complete (i.e. it admits any finite limit);

- $\mathcal{E}$ is cartesian closed, i.e. the functor $- \times B$ has a right adjoint $(-)^B$, for any $B \in \mathcal{E}$;

- The functor of subobjects $\mathrm{Sub}_{\mathcal{E}}(-)$ sending $A \in \mathcal{E}$ to the collection of (isomorphism classes) of monics $B \hookrightarrow A$ is representable by an object Ω, called *subobject (or truth) classifier*, in such a way that any $B \hookrightarrow A$ results as the pullback of a universal map $\mathrm{true}\colon * \to \Omega$, along an arrow $\chi_B\colon B \to A$:

$$\begin{array}{ccc} B & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle \mathrm{true}} \\ A & \xrightarrow[\chi_B]{} & \Omega. \end{array} \tag{7}$$

  The bijection

$$\mathrm{Sub}_{\mathbf{C}}(A) \cong \mathcal{E}(A,\Omega) \tag{8}$$

  is realized by the correspondence $[B \hookrightarrow A] \mapsto \chi_B$, the "characteristic map" of $B \subseteq A$.

EXERCISE 1 : *Define the functor* $\mathrm{Sub}_{\mathbf{C}}(-)$ *on arrows; show that* true *corresponds to* 1_Ω *via the bijection (**8**).*

EXERCISE 2 [**Sets** IS A TOPOS]: *The name for χ_B betrays the fact that the category* **Sets** *is a topos. In fact, it is an elementary topos (show that $\Omega = \{0,1\}$, and χ_B is exactly the characteristic function of $B \subseteq A$, sending B to* 1 *and $A \setminus B$ to* 0*).*

Show that **Sets** *is also a Grothendieck topos finding a Grothendieck site X such that* **Sets** $\cong \mathrm{Sh}(X)$.

COROLLARY 1 [**FSets** IS A TOPOS]: *Finite limits of finite sets are finite, and for any $A,B \in$* **FSets** *the set $B^A = \hom(A,B)$ has $|B|^{|A|}$ elements. Then any limit computed in* **Sets** *commutes with the forgetful functor $U\colon$* **FSets** $\hookrightarrow$ **Sets**, *hence $\Omega = \{0,1\}$ classifies* a fortiori *subsets of finite sets.*

Example 1.8 ($[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$ is a topos)**.** Completeness is obviously inherited by **Sets**. The only non-trivial problem is to define the exponential object G^F such that the bijection

$$\mathrm{Nat}(H \times F, G) \cong \mathrm{Nat}(H, G^F) \tag{9}$$

is natural both in H and in G. Suppose it exists, then by the Yoneda lemma we must have

$$G^F(A) \cong \mathrm{Nat}(\mathrm{Y}(A), G^F) \cong \mathrm{Nat}(\mathrm{Y}(A) \times F, G) \tag{10}$$

hence we *define* G^F to act on A in this precise way, and the category is automatically cartesian closed[4].

Again by Yoneda lemma, if $[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$ has a subobject classifier, then it must be defined as

$$\underline{\Omega}(A) \cong \mathrm{Nat}(\mathrm{Y}(A), \Omega) \cong \mathrm{Sub}(\mathrm{Y}(A)) \tag{11}$$

hence $\underline{\Omega}(A) := \{S \mid S \hookrightarrow \mathrm{Y}(A)\}$ The truth classifier is defined to be the natural transformation $\mathrm{Y}(*) \to \underline{\Omega}(A)\colon \{*\} \to \mathrm{Y}(A) = \mathbf{C}(-,A)$, the maximal subfunctor.

[4] This result can be deduced from a pure categorical argument involving bicompleteness of **Sets** and its cartesian closure: indeed it is possible to prove that the end $\int_Y \left(\prod_{f:Y\to C} G(Y)^{F(Y)}\right)$ is the desired exponential object applied to $C \in \mathbf{C}$.

EXERCISE 3 : *Unravel the details behind the definition of* $\underline{\Omega}(-)$*, showing that it actually represents the subobject functor in* $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$.

EXERCISE 4 : *Notice that* $\underline{\Omega}$ *can be really far from the constant presheaf on* $* \amalg *$*; what does* $\underline{\Omega}(X)$ *look like if* $\mathbf{C} = \mathrm{Ouv}(X)$ *is the category of open subsets of a topological space?*

Remark 1 ($[\mathbf{C}, \mathbf{Sets}]$ is a topos)**.** The same argument obviously holds in the case of the category of *co*presheaves $[\mathbf{C}, \mathbf{Sets}] = [(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}, \mathbf{Sets}]$.

Any Grothendieck topos is an elementary topos: this can be shown with a little effort in view of the preceding remarks (the subobject classifier of $\mathrm{Sh}(\mathbf{C}, \mathrm{COV})$ is the same as that in $[\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$, because $\underline{\Omega}$ is indeed a sheaf; exponential object are the same too. because if G is a sheaf, then so is G^F for any $F \in [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}]$). On the contrary, not every elementary topos is the topos of sheaves on a site: consider for example the topos of finite sets as in Example **1**.

In fact, despite the number of examples we gave, "being a topos" is rather a lucky situation, because the condition that $\mathrm{Sub}_{\mathbf{C}}(-)$ is representable is kind of a strict one.

Even categories which are really "tame" in other respects rarely happen to be toposes. For example

EXERCISE 5 : *Show that*

- **Grp** *is not an elementary topos;*
- *There is no non-trivial abelian category which is a topos;*
- *The category of compactly generated, Hausdorff topological spaces* **CGHaus** *is an elementary topos (find the right topology on its subobject classifier!), but* $\mathrm{Ab}(\mathbf{Top})$ *(the category of topological abelian groups, which in any case is not abelian) is not.*

1.3 Morphisms of topoi.

As you are probably aware, category theory is totally about *arrows*. Once defined the objects, we want to turn the collection of those objects into a category defining *morphisms*.

Morphisms of toposes come as suitable pairs of adjoint functors: the definition of such morphisms is modeled on the following paradigmatic example, which is well-known as a basic fact in sheaf theory.

Let X, Y be topological spaces and $f : X \to Y$ a continuous function. It is a truism that the pair of functions $f^{\leftarrow} : \mathrm{Ouv}(Y) \rightleftarrows \mathrm{Ouv}(X) : f_*$ defines a Galois connection between the posets $\mathrm{Ouv}(X), \mathrm{Ouv}(Y)$. In the same manner, it is a well-understood fact in sheaf theory that this Galois connection lifts to an adjunction between the categories $\mathrm{Sh}(X), \mathrm{Sh}(Y)$. We recall this, basically copying **[Schapira]**.

Definition 1.6 (Direct Image Sheaf)**.** Let $\mathcal{G} \in \mathrm{Sh}(X)$. Define the sheaf $f_*\mathcal{G}$ on Y as $U \mapsto \mathcal{G}(f^{\leftarrow}U)$, and for any $\varphi : \mathcal{G} \to \mathcal{G}'$ morphism of sheaves on X, $f_*\varphi : f_*\mathcal{G} \to f_*\mathcal{G}'$ is a morphism between sheaves on Y defined by $\varphi * f^{\leftarrow}$:

$$\begin{array}{ccc} \mathcal{G}(f^{\leftarrow}U) & \xrightarrow{\varphi_{f^{\leftarrow}U}} & \mathcal{G}'(f^{\leftarrow}U) \\ {\scriptstyle\text{res}}\big\downarrow & & \big\downarrow{\scriptstyle\text{res}} \\ \mathcal{G}(f^{\leftarrow}V) & \xrightarrow[\varphi_{f^{\leftarrow}V}]{} & \mathcal{G}'(f^{\leftarrow}V) \end{array} \tag{12}$$

Definition 1.7 (Inverse Image Sheaf)**.** Let $\mathcal{F} \in \mathrm{Sh}(Y)$, define $f^{-1}\mathcal{F}$ to be the sheaf on X associated to the presheaf

$$V \mapsto \varinjlim_{U \supseteq fV} \mathcal{F}(U) \tag{13}$$

Restriction are determined by the fact that (calling $\mathfrak{I}(V)$ the set of neighborhoods of $f(V)$ in Y) $W \subseteq V \Rightarrow \mathfrak{I}(fV) \subseteq \mathfrak{I}(fW)$; hence there exists $\varinjlim_{U \in \mathfrak{I}(V)} \mathcal{F}(U) \to \varinjlim_{U \in \mathfrak{I}(W)} \mathcal{F}(U)$.

Given a morphism between sheaves $\varphi \colon \mathcal{F} \to \mathcal{F}'$, the induced map between sheaves on X is simply determined by

$$\begin{array}{ccc} \varinjlim_U \mathcal{F}(U) & \xrightarrow{\varinjlim \varphi_U} & \varinjlim_U \mathcal{F}'(U) \\ {\scriptstyle\text{res}}\big\downarrow & & \big\downarrow{\scriptstyle\text{res}} \\ \varinjlim_U \mathcal{F}(U) & \xrightarrow[\varinjlim \varphi_U]{} & \varinjlim_U \mathcal{F}'(U) \end{array} \tag{14}$$

Remark 2. Let $\mathscr{R}$ be a sheaf of rings on X, $f^{-1}\mathscr{R} \in \mathrm{Sh}(Y)$. If $\mathscr{S}$ is a sheaf of rings on Y, then $f_*\mathscr{S} \in \mathrm{Sh}(X)$. These two correspondences induce functors

$$f_* \colon \mathrm{Mod}(\mathscr{S}) \to \mathrm{Mod}(f_*\mathscr{S})$$
$$f^{-1} \colon \mathrm{Mod}(\mathscr{R}) \to \mathrm{Mod}(f^{-1}\mathscr{R})$$

Remark 3. The two functors $f^{-1} \colon \mathrm{Sh}(Y) \rightleftarrows \mathrm{Sh}(X) \colon f_*$ are adjoints in such a way that $f^{-1} \dashv f_*$: counit and unit of the adjunction are respectively

$$\varepsilon \colon f^{-1} \circ f_* \to 1 \tag{15}$$

induced by the fact that $(f^{-1} \circ f_*\mathcal{G})(V) = \varinjlim_{W \supseteq fV} \mathcal{G}(f^{\leftarrow}W)$, and that if $W \supseteq fV$, then $f^{\leftarrow}W \supseteq V$ (as immediately follows from the chain of inclusions $f^{\leftarrow}W \supseteq f^{\leftarrow}fV \supseteq V$): this entails that for any $W \supseteq fV$ there exists $\mathcal{G}(f^{\leftarrow}W) \to \mathcal{G}(V)$, hence an arrow $(f^{-1} \circ f_*\mathcal{G})(V) = \varinjlim_{W \supseteq fV} \mathcal{G}(f^{\leftarrow}W) \to \mathcal{G}(V)$, and the transformation

$$\eta \colon 1 \to f_* \circ f^{-1} \tag{16}$$

induced by $(f_* \circ f^{-1}\mathcal{F})(U) = \varinjlim_{W \supseteq f(f^{\leftarrow}U)} \mathcal{F}(W)$, hence U contains $f(f^{\leftarrow}U)$: this entails that there exists a map $\mathcal{F}(U) \to (f_* \circ f^{-1}\mathcal{F})(U) = \varinjlim_{W \supseteq f(f^{\leftarrow}U)} \mathcal{F}(W)$.

In view of the previous paradigmatic example we are led to define a morphism of toposes in the following way

Definition 1.8 (Geometric Morphism)**.** Let $\mathcal{E},\mathcal{F}$ be (either elementary or Grothendieck) toposes. A *geometric morphism* $f\colon \mathcal{F}\to\mathcal{E}$ consists of a pair of adjoint functors

$$\mathcal{F} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathcal{E} \tag{17}$$

such that the left part f^* is left exact, i.e. it commutes with finite limits. The functor f^* is called the *left* or *inverse image part* of the morphism, and f_* is called the *right* or *direct image part*.

It is customary to denote the domain/codomain of a geometric morphism as the domain/codomain of its right part f_*.

Proposition 1.1. Suppose $\mathcal{F}=\mathrm{Sh}(X),\mathcal{E}=\mathrm{Sh}(Y)$ for two sober spaces X,Y. Then any geometric morphism $\mathcal{F}\to\mathcal{E}$ comes from a unique continuous function $f\colon X\to Y$.

Proof. See **[Moerdijk]**, pp. **348-349** for a proof in the case Y is T2. The proof of the general case is folklore but it is more involved.

Notice that this result means that for any $f^*\colon \mathcal{F}\rightleftarrows\mathcal{E}\colon f_*$ there exists a unique continuous map $\varphi\colon X\to Y$ such that, in the notations before,

$$\varphi^{\leftarrow}=f^*,\qquad f_*=\varphi_*. \qquad \square \tag{18}$$

Example 1.9. The category of presheaves on $\mathbf{C}$, $[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$, can be turned into a site in various ways; one of its most useful Grothendieck topologies is the finest for which every *representable* functor $\mathrm{Y}(C)=\hom(-,C)$ is a sheaf: it is called the *canonical* topology on $[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$.

The functor $i\colon \mathrm{Sh}(\mathbf{C},\mathrm{COV}_{\mathbf{C}})\hookrightarrow[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$ is known to admit a left adjoint[5], the *sheafification* functor $(-)^{\mathrm{a}}\colon F\mapsto F^{\mathrm{a}}$. This turns $\mathrm{Sh}(\mathbf{C},\mathrm{COV}_{\mathbf{C}})$ into a reflective subcategory of $[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$, and the pair of adjoint functors

$$\mathrm{Sh}(\mathbf{C},\mathrm{COV}_{\mathbf{C}}) \underset{(-)^{\mathrm{a}}}{\overset{i}{\rightleftarrows}} [\mathbf{C}^{\mathrm{op}},\mathbf{Sets}] \tag{19}$$

is a geometric morphism once the category $[\mathbf{C}^{\mathrm{op}},\mathbf{Sets}]$ is endowed with the canonical topology.

Example 1.10. Let $\mathcal{E}$ be a(n elementary) topos. For any object $B\in\mathcal{E}$ the slice category $\mathcal{E}/B$, whose objects are arrows $X\to B$ is again a topos (this is Theorem **IV.7.1** in **[Moerdijk]**). Any morphism $k\colon B\to B'$ induces a functor $\mathcal{E}/B'\to\mathcal{E}/B$ by pulling back $X\to B'$ along $B\to B'$, and we does expect that this defines (the left part of) a geometric morphism $\mathcal{E}/B\to\mathcal{E}/B'$. Indeed this is true, because k^* can be shown to admit both a left and a right adjoint, respectively $\sum_k$ and $\prod_k$ (this is **[Moerdijk]**, thm. **IV.7.2**), hence the adjunction $k^*\dashv\prod_k$ defines the desired morphism.

[5]This can be viewed either mimicking the topological construction of the case $\mathbf{C}=\mathrm{Ouv}(X)$ for some space X or better, by means of a purely categorical argument.

Definition 1.9. A morphism between toposes often happens to be geometric because its left part f^* admits a left adjoint $f_!$; these geometric morphisms are called *essential*.

Remark 4. Let $f\colon X \to Y$ be an étale morphism of schemes. This gives rise to a geometric morphism of topoi $f\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ which is always essential.

Indeed, if we denote as $\mathrm{\acute{E}t}(X)$, $\mathrm{\acute{E}t}(Y)$ the sites of sheaves with the étale topology, then the functor $f_!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)\colon p \mapsto f \circ p$ is both cocontinuous (**[SGA4]**, **III.2.1**) and continuous (*ibi*, **III.1.1**), and by *ibi*, **III.2.6** $f_!$ induces the string of adjoints

$$f_! \dashv f^* \dashv f_*, \tag{20}$$

hence an essential geometric morphism $f\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

A similar argument applies to the classical Grothendieck site $(\mathbf{Top}, \mathrm{COV}_{\mathrm{cl}})$: an open embedding $j\colon U \to Y$ always induces an essential geometric morphism $j\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X)$, where $j_! \dashv j^* \dashv j_*$ is defined to act as "extension by $\emptyset$": it suffices to sheafify the presheaf defined by

$$j_!\mathcal{F}(W) = \begin{cases} \mathcal{F}(W) & \text{if } W \subseteq U \\ \emptyset & \text{otherwise} \end{cases} \tag{21}$$

(it is worth to notice that **[Schapira]**, Proposition **2.3.6** shows that any other sheaf satisfying the same condition must be isomorphic to $j_!\mathcal{F}$).

1.4 The topos of group actions.

The following section is devoted to present a topic which is ubiquitous in both the following discussion about Joyal's strong stacks and in general topos theory. Despite the fact that toposes are rather rare structures, there exists a huge family of well-behaved categories obtained from a given topos $\mathcal{E}$, which are themselves toposes: these are categories of *G-objects* in $\mathcal{E}$, i.e. objects $X \in \mathcal{E}$ endowed with an action of an *internal group* $G \in \mathrm{Grp}(\mathcal{E})$.

It is completely straightforward to define such a notion in the case $\mathcal{E} = \mathbf{Sets}$, where G-objects are nothing more than G-sets in the classical sense, morphisms are equivariant maps $f\colon X_G \to Y_G$ such that $f(g.x) = g.f(x)$ (i.e. they commute with the action $\alpha\colon G \times X \to X\colon x \mapsto g.x$). The only subtlety is, as usual, to find the representative for the functor of subobjects; what is a "sub-G-set" of a G-set X_G?

We now want to extend the notion of G-action to the case of an object $X \in \mathcal{E}$: to this end we need to get acquainted with the notion of *internalization* of an algebraic structure in a (finitely complete) category.

1.5 Internalization: monoids, groups, categories.

It is straightforward to notice that given a monoid $(M, \cdot, 1)$, the associative property of the multiplication $m\colon M \times M \to M$ and the fact that $m(a,1) = a = m(1,a)$ can be expressed in a purely diagrammatical way, via the arrows $m\colon M \times M \to M$ and $e\colon \{*\} \to M$ ($\{*\}$ is a particular choice of a terminal object in **Sets**) in the diagrams

$$\begin{array}{ccc} M\times M\times M & \xrightarrow{\mathrm{id}_M\times m} & M\times M \\ {\scriptstyle m\times\mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle m} \\ M\times M & \xrightarrow[m]{} & M \end{array} \qquad \begin{array}{ccccc} \{*\}\times M & \xrightarrow{e\times\mathrm{id}_M} & M\times M & \xleftarrow{\mathrm{id}_M\times e} & M\times\{*\} \\ & \searrow & \downarrow{\scriptstyle m} & \swarrow & \\ & & M & & \end{array} \tag{22}$$

Definition 1.10 (Monoid in **C**)**.** A *monoid* in a finitely complete category **C** consists of a triple $(M, m\colon M\times M\to M, e\colon 1\to M)$ such that diagrams in (**22**) commute.

We often abuse notations and call *monoid* the object M alone.

Definition 1.11 (Category of monoids in **C**)**.** Monoids (M, m_M, e_M) can be arranged to form a category whose morphisms $(M, m_M, e_M)\to(N, m_N, e_N)$ are arrows $h\colon M\to N$ in **C** which "commute with multiplication and respect identities".

Remark 5. As for the notion of structure, the notion of *action* of a structured set on another set can be generalized to the notion of action on an object, once we translated it into a diagram to be valid in any finitely-complete category **C**.

Suppose (M, m, e) is a monoid in such a **C**; then an action of M on an object $S\in\mathbf{C}$ consists of an arrow $a\colon M\times S\to S$ such that the diagram

$$\begin{array}{ccc} M\times M\times S & \xrightarrow{\mathrm{id}_M\times a} & M\times S \\ {\scriptstyle m\times\mathrm{id}_S}\downarrow & & \downarrow{\scriptstyle a} \\ M\times S & \xrightarrow[a]{} & S \end{array} \qquad \begin{array}{ccc} 1\times S & \xrightarrow{e\times\mathrm{id}} & M\times S \\ & \searrow & \downarrow{\scriptstyle a} \\ & & S \end{array} \tag{23}$$

commute. The first equality amounts to ask that $h.(g.x) = (h\circ g).x$; the second is the request that $e_G.x = x$ for any $x\in S$.

EXERCISE 6 : *Let* **C** *be a category. A* monad *consists of a monoid in the category of endofunctors* $[\mathbf{C},\mathbf{C}]$.

- *Unravel the definition of a monad in terms of a functor $T\colon \mathbf{C}\to\mathbf{C}$ endowed with two natural transformations $\mu\colon T\circ T\to T$, $\eta\colon 1\to T$ (1 is the identity functor of* **C***) subject to suitable commutativity conditions;*
- *Show that any pair of adjoint functors $\langle F\dashv G,\eta,\varepsilon\rangle$ gives rise to a monad where $T = GF$, the multiplication μ is defined to be the natural transformation $G\varepsilon F\colon G(FG)F\to GF$, and the unit $\eta\colon 1\to GF$ is precisely the unit of the adjunction.*

1.5.1 Internal groups and rings.

Once we understood how to "categorify" the notion of monoid, an analogous procedure allows to define groups and rings internal to a finitely complete category: suitable diagrams will translate via *categorical semantics* additional operations, their properties and morphisms between them, defining Grp(**C**), and Rng(**C**).

Definition 1.12 (Internal group)**.** Let $\mathbf{C}$ be a finitely complete category. A *group* in $\mathbf{C}$ consists of a monoid $(G,m,1) \in \mathrm{Mon}(\mathbf{C})$ endowed with an arrow $i\colon G \to G$ (*inversion*) such that $i \circ i = \mathrm{id}_G$ and such that the diagram

$$\begin{array}{ccccc} G\times G & \xleftarrow{\Delta} & G & \xrightarrow{\Delta} & G\times G \\ \downarrow{\scriptstyle 1_G\times i} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle i\times 1_G} \\ G\times G & \xrightarrow[m]{} & G & \xleftarrow[m]{} & G\times G \end{array} \tag{24}$$

commutes, where we denoted as $\Delta = \langle 1_G, 1_G \rangle$ the *diagonal morphism* $G \to G \times G$.

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \downarrow{\scriptstyle i} & b & \downarrow{\scriptstyle i} \\ G & \xrightarrow[h]{} & H \end{array}$$

Definition 1.13 (Group morphism in $\mathbf{C}$)**.** A morphism $h\colon G \to H$ between groups consists of a morphism of monoids which commutes with i, in the sense that the diagram aside commutes.

EXERCISE 7 : *Define an* internal ring *in a finitely complete category* $\mathbf{C}$*; define a morphism of internal rings in such a way that* $Rng(\mathbf{C})$ *becomes a subcategory of* $\mathbf{C}$.

1.6 G-objects form a topos.

The main theorem of this section can be found in **[Moerdijk]**, **V.6.1**.

Theorem 1.2. Let $\mathcal{E}$ be a topos, and G an internal group in $\mathcal{E}$. Then the category $\mathcal{E}^G$ of G-objects in $\mathcal{E}$ is again a topos.

Notice that in the case $\mathcal{E} = \mathbf{Sets}$ the result is a direct corollary of Example **1.8** and Remark **1**, because any group can be regarded as a category $\mathbf{G}$ with a single object $\odot$ such that $\mathbf{G}(\odot,\odot) \cong G$; then the category of right/left G-sets simply is the category of covariant/contravariant functors $\mathbf{G} \to \mathbf{Sets}$.

Sketch of Proof. The forgetful functor $U\colon \mathcal{E}^G \to \mathcal{E}$ which sends $(X, a\colon G \times X \to X)$ to X is monadic (its left adjoint is the free-G-action functor, defined by sending X in $(A = G \times E, m \times 1_E\colon G \times A \to A)$; moreover it creates every limit in $\mathcal{E}^G$ (**[Moerdijk]**, prop. **IV.4.1**), hence $\mathcal{E}^G$ is complete.

The action on the exponential object C^B of two G-objects $B, C \in \mathcal{E}$ is given by (the transpose of) the arrow

$$G \times C^B \times B \xrightarrow{\Delta \times C^B \times B} G \times G \times C^B \times B \xrightarrow{\cong} G \times C^B \times G \times B \xrightarrow{G \times C^B \times a_B}$$
$$\xrightarrow{G \times C^B \times a_B} G \times C^B \times B \xrightarrow{G \times \mathrm{ev}} G \times C \xrightarrow{a_C} C$$

The subobject classifier is inherited by that of $\mathcal{E}$, once we turn true$\colon * \to \Omega$ into a G-map; this can be done giving both $*$ and Ω the trivial action $\pi_2\colon G \times \Omega \to \Omega$. $\square$

1.7 Internal categories.

A finitely complete category allows one to internalize... even the notion of a category! For the sake of clarity we fix once and for all a Grothendieck topos $\mathcal{E}$ to work in.

Definition 1.14. An *internal category* in $\mathcal{E}$ consists of the following arrangement $\mathbb{C} = (C_o, C_a, s, t, c, e)$ of objects and arrows in $\mathcal{E}$:

- A pair of objects $C_o, C_a \in \mathcal{E}$, respectively the *object of objects* and the *object of arrows*;
- Arrows $s, t \colon C_a \to C_o$ (source and target), $e \colon C_o \to C_a$ (identity), $c \colon C_a \times_{C_o} C_a \to C_a$ (composition, where $C_a \times_{C_o} C_a$ = "composable arrows" is defined to be the pullback along s, t) such that:
  - "the source and the target of the identity map coincide", i.e. the following diagram commutes:

$$\begin{array}{ccccc} C_o & \xrightarrow{e} & C_a & \xleftarrow{e} & C_o \\ & \diagdown & {\scriptstyle s}\downdownarrows{\scriptstyle t} & \diagup & \\ & & C_o & & \end{array} \tag{25}$$

  - "the source of $g \circ f$ is the source of f, and the target of $g \circ f$ is the target of g", i.e. the following square commutes either if we choose the inner or the outer arrows:

$$\begin{array}{ccc} C_a \times_{C_o} C_a & \overset{c}{\underset{c}{\rightrightarrows}} & C_a \\ {\scriptstyle p_1}\downdownarrows{\scriptstyle p_2} & & {\scriptstyle t}\downdownarrows{\scriptstyle s} \\ C_a & \overset{t}{\underset{s}{\rightrightarrows}} & C_o \end{array} \tag{26}$$

  - "composition of arrows is associative", i.e. the following diagram commutes:

$$\begin{array}{ccc} C_a \times_{C_o} C_a \times_{C_o} C_a & \xrightarrow{c \times_{C_o} 1} & C_a \times_{C_o} C_a \\ {\scriptstyle 1 \times_{C_o} c}\downarrow & & \downarrow{\scriptstyle c} \\ C_a \times_{C_o} C_a & \xrightarrow[c]{} & C_a \end{array} \tag{27}$$

  - "e acts like an identity for the composition", i.e. the following diagram commutes:

$$\begin{array}{ccccc} C_o \times_{C_o} C_a & \xrightarrow{e \times_{C_o} C_a} & C_a \times_{C_o} C_a & \xleftarrow{C_a \times_{C_o} e} & C_a \times_{C_o} C_o \\ & {\scriptstyle p_2}\searrow & \downarrow{\scriptstyle c} & \swarrow{\scriptstyle p_1} & \\ & & C_a. & & \end{array} \tag{28}$$

Remark 6. Internal categories can be characterized in various other ways:

- (see **[Johnstone]**, Remark **2.13** and **[Cisinski]**, p. **19**) One can generalize the categorial *nerve-realization paradigm* [6] to the setting of internal categories in $\mathcal{E}$, once noticed that the "classical" nerve is a suitable simplicial set[7] functorially associated to $\mathbb{C} \in \text{Cat}(\mathcal{E})$ in the case $\mathcal{E} = \textbf{Sets}$. Define $N(\mathbb{C}) \in [\Delta^{\text{op}}_{\leq 3}, \mathcal{E}]$, where $\Delta_{\leq n}$ is the full subcategory of Δ with objects $\{[0], \dots, [n]\}$, to be the simplicial $\mathcal{E}$-object having as object of n-simplices the n-fold pullback of C_a along s, t: $N(\mathbf{C})_0 = C_o$, $N(\mathbf{C})_1 = C_a$, $N(\mathbf{C})_2 = C_a \times_{C_o} C_a$, $N(\mathbf{C})_3 = C_a \times_{C_o} C_a \times_{C_o} C_a \dots$

  The perseverant reader is invited to check that faces and degeneracy maps are induced exactly by the source, target, indentity, and composition arrows. The nerve functor defined in this way is fully faithful, hence the category $\text{Cat}\,\mathcal{E}$ of internal categories in $\mathcal{E}$ can be faithfully identified with a full subcategory of truncated simplicial sets, $[\Delta^{\text{op}}_{\leq 3}, \mathcal{E}]$.

- (see **[Betti]**) Define the (2-)category $\text{Span}\,\mathcal{E}$ of *spans* in $\mathcal{E}$ having the same objects of $\mathcal{E}$, and where an arrow $X \dashrightarrow Y$ consists of a *roof* $X \leftarrow A \to Y$; any hom-set in $\text{Span}\,\mathcal{E}$, say between X and Y, is in addition a category if we define 2-cells to be $\eta : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & \searrow^{\eta} & \uparrow \\ X & \longleftarrow & B. \end{array} \tag{29}$$

  Now it's easily seen that internal categories in $\mathcal{E}$ bijectively correspond to *monads* in $\text{Span}\,\mathcal{E}$ (see Exercise **6** for the definition of a monad).

Definition 1.15 (Internal functor)**.** If $\mathbb{C} = (C_o, C_a, s, t, c, e)$ and $\mathbb{D} = (D_o, D_a, s', t', c', e')$ are internal categories in some ambient category $\mathcal{E}$, then an *internal functor* consists of two

[6] This is a rather *folkloristic* topic in Category Theory: it can be stated as follows: *any functor* $\gamma\colon \mathbf{C} \to \mathbf{D}$ *to a cocomplete category induces an adjoint pair* $\rho : \widehat{\mathbf{C}} \leftrightarrows \mathbf{D} : N$, and deduced from the following argument, which we found for the first time in **[Dugger]**: the functor ρ (the "geometric realization") is the left Yoneda extension of γ, $\text{Lan}_{\text{Y}}(\gamma)$, computed as the coend $\int^C \widehat{\mathbf{C}}(\text{Y}(C), -) \pitchfork \gamma(C)$ (recall that $\mathbf{D}$ is cocomplete, hence cotensored over **Sets**), and its right adjoint N (the "nerve") is defined since we have the following chain of isomorphisms:

$$\mathbf{D}\big(\text{Lan}_{\text{Y}}\gamma(F), D\big) \cong \mathbf{D}\left(\int^C \widehat{\mathbf{C}}(\text{Y}(C), F) \pitchfork \gamma(C), D\right) \cong \mathbf{D}\left(\int^C F(C) \pitchfork \gamma(C), D\right) \cong$$
$$\cong \int_C \mathbf{D}(F(C) \pitchfork \gamma(C), D) \cong \int_C \textbf{Sets}(FC, \mathbf{D}(\gamma(C), D)) \cong \widehat{\mathbf{C}}(F, \mathbf{D}(\gamma(-), D)).$$

[7] A simplicial set can be thought as a graded set $\{K_n\}$ with suitable functions $\partial^n_j\colon K_n \to K_{n-1}$ (*faces*), $s^n_j\colon K_n \to K_{n+1}$ (*degeneracies*) satisfying suitable *simplicial identities* (which can be deduced from the fact that a simplicial set is nothing but a presheaf over $\Delta = \textbf{FinOrd}$, the category of totally ordered nonempty finite sets and monotone maps between them: see **[Mac Lane]**, §**7.5**; similar identities hold in Δ, hence by functoriality they hold in $[\Delta^{\text{op}}, \textbf{Sets}]$). The category Δ is *generated* by face and degeneracy arrows, modded out by simplicial identities.

arrows $F_o\colon C_o \to D_o$, $F_a\colon C_a \to D_a$ which turn suitable diagrams into commutative ones:

$$\begin{array}{ccc} C_a & \xrightarrow{F_a} & D_a \\ {\scriptstyle t}\downarrow\uparrow\downarrow{\scriptstyle s} & & {\scriptstyle t'}\downarrow\uparrow\downarrow{\scriptstyle s'} \\ C_o & \xrightarrow[F_o]{} & D_o \end{array} \qquad \begin{array}{ccc} C_a \times_{C_o} C_a & \xrightarrow{F_a \times_{F_o} F_a} & D_a \times_{D_o} D_a \\ {\scriptstyle c_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle c'} \\ C_a & \xrightarrow[F_a]{} & D_a \end{array} \tag{30}$$

(the left one commutes anytime you choose as vertical arrows either both s, t or the central arrow e.) These diagrams obviously express

- The fact that a functor respects source and target of any arrow;
- The fact that a functor respects composition of arrows and identities.

Definition 1.16 (Internal Natural Transformation)**.** Given $\mathbb{C} = (C_o, C_a, s, t, c, e)$, $\mathbb{D} = (D_o, D_a, s', t', c', e') \in \mathit{Cat}(\mathcal{E})$, and internal functors $F, G\colon \mathbb{C} \to \mathbb{D}$, an *internal natural transformation* consists of an arrow $\alpha\colon C_o \to D_a$ such that

- $s' \circ \alpha = F_o$, $t' \circ \alpha = G_o$
- $c' \circ (\alpha \circ t, F_a) = c' \circ (G_a, \alpha \circ s)$

where $(\alpha \circ t, F_a)$ and $(G_a, \alpha \circ s)$ are easily seen to be composable with the composition arrow $c\colon C_a \times_{C_o} C_a \to C_a$, as they are arrows $C_a \rightrightarrows D_a$ such that $s' \circ \alpha \circ t = F_o \circ t = t' \circ F_a$ and $s' \circ G_a = t' \circ \alpha \circ s$.

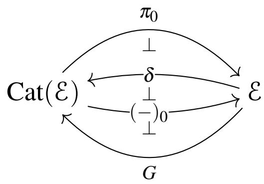
Internal functors compose in the obvious way; internal natural transformation compose in two different ways (horizontally, i.e. *à la façon de Godement*, and vertically, as every pair of 2-cells do in a 2-category: see **[Borceux]**, §**7**, **8**), as they do in the case $\mathcal{E} = \mathbf{Sets}$, so the collection $\mathrm{Cat}(\mathcal{E})$ of internal categories, functors and natural transformations in $\mathcal{E}$ is the prototype of an *internal 2-category* in $\mathcal{E}$ (but we do not intend to begin the slippery climp of higher category theory, even more in the setting on *internal* categories! We address the interested reader to the short and neat paper by **[Betti]**).

As a final remark, we want to propose an internalization of a classical result valid when $\mathcal{E} = \mathbf{Sets}$, which can be found in **[Gray]**. In a few words there exists a string of adjunctions

$$\pi_0 \dashv \delta \dashv (-)_o \dashv G \tag{31}$$

where the functor $\pi_0\colon \mathbf{Cat} \to \mathbf{Sets}$ sends a category to its set of *connected components* obtained as the coequalizer of the pair $\mathrm{src}, \mathrm{trg}\colon \hom(\mathbf{C}) \to \mathrm{Ob}(\mathbf{C})$, δ is the functor sending a set to the discrete category on it, $(-)_o$ is the functor which sends a (small) category to its set of objects, and G sends a set X to the maximally connected groupoid on it, obtained taking X as set of objects and choosing exactly *one* isomorphism between any two elements of X.

This result can be easily internalized obtaining a similar string of adjunctions



(32)

The discrete category functor can be defined as follows: an object $X \in \mathcal{E}$ is sent to the category $\delta X = (X, X, 1, 1, \Delta)$, where source, target and composition (once identified $X \times_X X \cong X$) are all identity arrows. This choice entails that an arrow $X \to C_o$ corresponds exactly to a functor $\delta(C) \to \mathbb{C}$: giving such a functor $\delta(X) \to \mathbb{C}$ means we are given arrows (F_o, F_a) such that $s_C \circ F_a = t_C \circ F_a = F_o$ and $e_C \circ F_o = F_a$ and such that $\Delta \circ F_a \times F_a = F_a$; on the other hand to build such a functor it's enough to know its object part F_o, since F_a must be equal to $e_C \circ F_o$.

To define the functor G it's enough to internalize the condition for which any two objects admit exaclty an isomorphism. This can be done asking that $(s,t)\colon C_a \to C_o \times C_o$ is an isomorphism in $\mathcal{E}$. The adjunction $\mathcal{E}(G_o, A) \cong \mathbf{Gpd}(\mathcal{E})(\mathbb{G}, G(A))$ is obtained taking the object-part of functors $F = (F_o, F_a)\colon \mathbb{G} \to G(A)$, and conversely, given $F_o\colon G_o \to A$ we have only one choice to define a functor F on arrows, according to how we defined $G(A)$. To check that this correspondences compose to the identity, it's enough to recal that the isomorphism between any two objects is *unique*.

2 Model categories.

Quillen model categories are "convenient categories to do homotopical algebra in", and to view them as *non-abelian counterparts of Grothendieck abelian categories*.

Tibor Beke

INTRODUCTION. A *model category* is a category endowed with three suitably interacting classes of morphisms, *weak equivalences*, *fibrations* and *cofibrations*, that allow us to build Homotopy Theory(ies) in a purely arrow-theoretic setting.

The definition of model categories as an abstract setting for homotopy theory is due to [**Quillen**]'s seminal work (even if a tentative "Abstract homotopy theory" dates back to Kan's series of articles on simplicial homotopy published since 1956), and the philosophy behind that definition is

> Thou shalt astray a minimal set of properties that permit to extend homotopy theory to categories other than topological spaces; moreover, thou shalt try to *internalize* classical homotopy-theoretical notions (the theory of fundamental groups and higher homotopy groups, stable homotopy, action of the π_1 on the fibers of a space, the behaviour of a covering map with respect to paths and homotopies, ...) in a suitable "category with weak equivalences".

Roughly speaking, a weak equivalence in a category **C** is an arrow in a certain sense "as similar as possible" to an isomorphism (in classical homological algebra there exists a well-established notion of *quasi-isomorphism*), and what we want to do is to pass in a setting (the *homotopy category* of **C**, $\mathrm{Ho}(\mathbf{C})$) where this arrow is a real isomorphism, adding the inverse it lacks: this apparatus willingly resembles the notion of (weak) homotopy equivalence in algebraic topology, where such maps are continuous functions $f\colon X \to Y$ inducing isomorphisms between all homotopy groups. The purely formal procedure of inversion of all quasi-isomorphisms falls under the name of *localization theory*, and it has been introduced by **[Zisman]** in their famous book: weak equivalences are all we need, or in a few words

all that matters is what we want to invert,

in the sense that *any* category with a distinguished class of weak equivalences can be endowed with an "homotopical calculus" which allows us to define *homotopy invariants of objects*. The whole machinery gravitating around weak equivalences serves in fact only to avoid certain annoying pathologies: fibrations and cofibrations work in sinergy ensuring that the localized category $\mathrm{Ho}(\mathbf{C}) =: \mathbf{C}[\mathrm{WK}^{-1}]$ is not as badly-behaved as it might happen (set-theoretic issues can prevent $\mathrm{Ob}(\mathrm{Ho}(\mathbf{C}))$ from being a set). They also ensure that we can figure the -highly untractable- set $\mathrm{Ho}(\mathbf{C})(A,Y)$ of arrows between A and Y in the localized category to be the set (and even before, to be *a* set) of (abstract) *homotopy classes* of arrows between A and Y.

In a few words, a model category consists of a 4-uple $(\mathbf{C}, \mathrm{WK}, \mathrm{FIB}, \mathrm{COF})$, where $(\mathbf{C}, \mathrm{WK})$ is a category with a distinguished class of weak equivalences, and $\mathrm{FIB}, \mathrm{COF}$ are two additional classes of arrows, the elements of which are called *fibrations* and *cofibrations*, having mutual *stability* and *lifting* properties. The leading principle behind homotopical algebra is that these properties give a tractable, albeit reasonably general, way to set up the basic machinery of homotopy theory in categories other than "spaces".

Definition 2.1 (Category with Weak Equivalences)**.** A *category with weak equivalences* is a category **C** with a distinguished class of morphisms $\mathrm{WK} \subseteq \mathrm{Mor}(\mathbf{C})$ which contains all isomorphisms of **C**, which is closed under composition and which satisfies the *two-out-of-three property*:

> For f,g any two composable morphisms of **C**, if any two of $\{f, g, g \circ f\}$ are in WK, then so is the third.

Definition 2.2 (Model Category)**.** A *(Quillen) model category* is a small-complete and small-cocomplete category endowed with three distinguished classes of morphisms: *weak equivalences*, WK; *fibrations*, FIB; *cofibrations*, COF, such that the following axioms are satisfied:

- $(\mathbf{C}, \mathrm{WK})$ is a category with weak equivalences;
- $\mathrm{WK}, \mathrm{FIB}, \mathrm{COF}$ are stable under taking *retracts*. Explicitly, the requirement that f is a retract of g means that there exist arrows i, j, u and v, such that the following

diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{i} & \mathbf{A}' & \xrightarrow{u} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{j} & \mathbf{B}' & \xrightarrow{v} & B \end{array} \qquad (u \circ i = 1,\ v \circ j = 1) \tag{33}$$

The condition of *stability* under retracts reads: if g has a retract f and it belongs to $\mathrm{WK}, \mathrm{FIB}, \mathrm{COF}$ then so does f.

$$\begin{array}{ccc} X & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & W, \end{array}$$

- For any commutative square like the one besides, where $i \in \mathrm{COF}$ and $p \in \mathrm{FIB}$, If either i or p is acyclic (i.e. it also belongs to WK), then there exists a lifting $Y \to Z$. In other words, acyclic fibrations/cofibrations have the *right/left lifting property* (RLP, LLP for short) with respect to fibrations/cofibrations;

- $(\mathrm{WK} \cap \mathrm{FIB}, \mathrm{COF})$, $(\mathrm{FIB}, \mathrm{WK} \cap \mathrm{COF})$ are (weak) factorization systems in $\mathbf{C}$, i.e. any arrow can be either factored as the composition of an acyclic fibration and a cofibration or as the composition of an acyclic cofibration and a fibration (*weakness* means that the factorization is not supposed to be unique).

Remark 7. Mutual lifting properties are what really define the notion of model category: a model category is uniquely determined by the datum of weak equivalences and fibration *or* by the datum of weak equivalences and cofibrations: in the first case, cofibrations are maps having the LLP with respect to acyclic fibrations, and in the second case fibrations are maps having the RLP with respect to acyclic cofibrations (see Proposition **3.13** in **[Dwyer-Spalinski]**).

Examples of model categories live in algebraic, topological and even pure-categorical contexts. Refer either to **[Hovey]** or again to **[Dwyer-Spalinski]** to have plenty of examples and explicit constructions, e.g. the fact that

- Topological spaces, homotopy equivalences and Serre fibrations/cofibrations form a model category;

- Simplicial sets, (simplicial) homotopy equivalences and Kan fibrations/cofibrations form a model category;

- For a given unitary ring R, chain complexes of R-modules, *quasi-isomorphisms* as weak equivalences and *degree-wise epimorphisms* as fibrations define a model category if we choose as COF exactly the class of maps in $\mathrm{LLP}(\mathrm{WK} \cap \mathrm{FIB})$.

- There are exactly *nine* model structures on the category of sets and functions[8].

Remark 8. We collect in a single paragraph various useful notational remarks.

[8] http://www.math.harvard.edu/~oantolin/notes/modelcatsets.html.

- Fibrations and cofibrations are denoted, especially in plotting commutative diagrams, as "injective" and "surjective" arrows: a cofibration is denoted $A \hookrightarrow B$, and a fibration $X \twoheadrightarrow Y$. This is not so astonishing as it is inspired by a paradigmatic example: try to unravel a model structure on **Sets** where the fibrations are precisely the surjective maps of sets.

- An object is said to be *(co)fibrant* if the unique arrow $(\emptyset \to)X \to 1$ is a (co)fibration. If it is the case (as it happens in the case of groupoids, cf. Remark **10**) that the process of factorization of an arrow as the composition of an acyclic fibration and a cofibration is *functorial*, we can define a functor using the factorization $\varnothing \to X$ as $\varnothing \to QX \overset{\sim}{\twoheadrightarrow} X$, sending X to $QX \overset{\sim}{\twoheadrightarrow} X$; this functor is called *fibrant replacement*.

- Define a completely dual notion of *cofibrant replacement* $X \overset{\sim}{\hookrightarrow} RX$.

It is again **[Dwyer-Spalinski]** which says "each of these settings has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language.

What is the *suspension* of an augmented commutative algebra? One of incidental appeals of Quillen's theory (to a topologist!) is that it both makes a question like this respectable and gives it an interesting answer."

Because of this, model categories and homotopical algebra can be seen not only as a branch of mathematics, but also as a useful tool to unravel mutual connection between various fields of modern research. In spite of the extreme neatness (which is nothing but the result of Quillen's striving) of the axioms defining it, the task of proving that a particular choice of weak equivalences and (co)fibrations really gives a model structure is often extremely long and involved: see for example **[Gelfand-Manin]**, **V.1.2-V.2.4** to get acquainted with the model structure on the category of simplicial sets.

Instead of a systematic presentation of model category theory and homotopical algebra, we prefer to propose a detailed (read: really pedantic) proof of a paradigmatic (as well as useful for the following discussion) example of model structure on a category: the so-called "folk model structure" on the (2-)category **Gpd** of groupoids.

2.1 The model structure on groupoids.

Recall that a *groupoid* is a category where every arrow is invertible. Groupoids become a (full sub)category (of **Cat**), denoted **Gpd**, if we define arrows $\mathbf{G} \to \mathbf{H}$ to be functors, and a 2-category if we allow natural transformation to turn every $[\mathbf{G},\mathbf{H}]$ into a category. The functor $i\colon \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ is both a left and a right adjoint: for example its right adjoint sends a category $\mathbf{C}$ to its *maximal subgroupoid* $\mathcal{G}(\mathbf{C})$, in such a way that $\mathbf{Cat}(\mathbf{G},\mathbf{C}) = \mathbf{Cat}(i\mathbf{G},\mathbf{C}) \cong \mathbf{Gpd}(\mathbf{G},\mathcal{G}(\mathbf{C}))$.

Theorem 2.1 (Folk model structure on **Gpd**)**.** The category **Gpd** admits a model structure where

- Weak equivalences are equivalences of groupoids;

- Cofibrations are functors which are injective on objects;

- Fibrations are *Grothendieck fibrations*, i.e. functors $p\colon \mathbf{E} \to \mathbf{B}$ such that for any $e \in \mathbf{E}$ and $\gamma\colon b \to p(e)$ there exists some $\eta : e' \to e$ such that $p(\eta) = \gamma$:

$$\begin{array}{ccc} {}^{\exists}e' & \longrightarrow & e \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ b & \longrightarrow & p(e). \end{array}$$

Proof. It's easy to prove that WK is a class of weak equivalences, because if we suppose that $f\colon \mathbf{E} \to \mathbf{G}, g \circ f\colon \mathbf{E} \to \mathbf{G} \to \mathbf{H}$ are equivalences of categories, then there are arrows

$$\begin{aligned} h\colon \mathbf{G} \to \mathbf{E} \ \text{ such that } fh \cong 1,\ hf \cong 1 \\ k\colon \mathbf{H} \to \mathbf{E} \ \text{ such that } gfk \cong 1,\ kgf \cong 1. \end{aligned}$$

Now it must be $(fk)g = f(kg) = fh \cong 1$, hence g is itself an equivalence of categories $\mathbf{G} \to \mathbf{H}$.

WK, FIB, COF are stable under retracts. Let's consider a diagram shaped like

$$\begin{array}{ccccc} \mathbf{X} & \xrightarrow{i} & \mathbf{X}' & \xrightarrow{u} & \mathbf{X} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle h}\big\uparrow\big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ \mathbf{Y} & \xrightarrow[j]{} & \mathbf{Y}' & \xrightarrow[v]{} & \mathbf{Y} \end{array} \tag{34}$$

where $u \circ i = 1_{\mathbf{X}}$, $v \circ j = 1_{\mathbf{Y}}$, then we have to show that if $g \in \{\text{WK}, \text{FIB}, \text{COF}\}$, then $f \in \{\text{WK}, \text{FIB}, \text{COF}\}$. Now:

- If g is a weak equivalence, then there exists $h\colon \mathbf{Y}' \to \mathbf{X}'$ such that $gh \cong 1, hg \cong 1$; hence uhj is the quasi=inverse of f, because $fuhj = vghj \cong vj = 1_{\mathbf{Y}}$, and $uhjf \cong ui = 1_{\mathbf{X}}$.

- If g is a fibration, and we are given an arrow $y \to f(x)$, then maps it in $j(y) \to jf(x) = gi(x)$, so that there exists x' such that $j(y) = g(x')$, i.e. $y = vg(x') = f(u(x'))$.

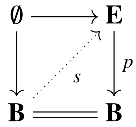
- If g is a cofibration, it means it is injective on objects; hence $f(x) = f(y)$ implies that $jf(x) = jf(y)$, i.e. $gi(x) = gi(y)$, $i(x) = i(y) \Rightarrow x = y$.

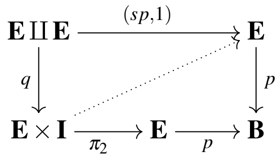
Anodyne fibrations. The rest of the proof is the difficult part: to acquire more agility, we define an auxiliary notion which turns out to be an equivalent characterization of trivial fibrations $p \in \text{WK} \cap \text{FIB}$.

Definition 2.3. We call $p\colon \mathbf{E} \to \mathbf{B}$ an *anodyne fibration* if it has the RLP with respect to *any* cofibration.

Proposition 2.1. Anodyne fibrations are exactly surjective-on-objects weak equivalences, i.e. all and only acyclic fibrations.

Proof. The arrow $\emptyset \to \mathbf{B}$ is a cofibration, hence the square besides commutes and has a filler, which is to say that p is surjective on objects. If we let $\mathbf{I} = \{0 \cong 1\}$ be the "interval" groupoid, we can consider the diagram

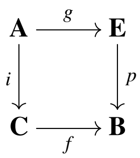




(35)

The arrow q which sends $(e,0)$ (in the coproduct) in $(e,0)$ in the product is a cofibration, hence there exists the dotted lifting u. Such an u serves to define a quasi inverse to p, because $uq(e,0) = sp(e), uq(e,1) = e$, and the fact that $(e,0)$ is isomorphic to $(e,1)$ in $\mathbf{E} \amalg \mathbf{E}$ implies that $sp \cong 1$, hence p is an equivalence of categories.

Conversely, if p is a surjective-on-objects equivalence of categories and $\mathbf{A} \xrightarrow{i} \mathbf{C}$ is a cofibration, the commutative diagram



(36)

induces an analogous diagram between the set of objects of the categories involved. In this diagram there is a diagonal filler

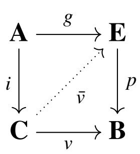
$$\overline{f_0} : C_0 \to E_0 = \begin{cases} sf_0 & \text{on } C_0 \setminus i(A_0) \\ g_0 & \text{on } i(A_0) \end{cases} \tag{37}$$

Suppose now $c \to c'$ is an arrow in $\mathbf{C}$; then $f(c \to c') = p(e \to e')$ for some $e \to e'$ which is unique because p is fully faithful. Nothing is left to do except defining $\bar{f}(c \to c') = e \to e'$. $\square$

Remark 9. Proving that anodyne fibrations coincide with maps in $\mathrm{WK} \cap \mathrm{FIB}$ is a matter of unraveling definitions: if $p \colon \mathbf{E} \to \mathbf{B}$ is anodyne it is a weak equivalence; given $\theta \colon b \to p(e)$, surjectivity on objects gives e' such that $p(e') = b$, and functoriality of Π_0 (or if you prefer, the fact that p is fully faithful), implies that there exist $\alpha \colon e' \to e$ such that $p(\alpha) = \theta$.

Conversely, if $p \in \mathrm{WK} \cap \mathrm{FIB}$ then the fibration condition implies that given $b \in \mathbf{B}$ and $b \to p(e)$ (there exists at least one,b ecause p is essentially surjective), there is e' such that $p(e') = b$, hence p is anodyne.

Lifting properties. First of all notice that p is anodyne iff it lifts any cofibration, iff it is in $\mathrm{WK} \cap \mathrm{FIB}$, hence in any square



(38)

where $p \in \text{WK} \cap \text{FIB}$ and $i \in \text{COF}$ there exists a diagonal filler.

Consider now the same commutative square, but now $i \in \text{COF} \cap \text{WK}$, $p \in \text{FIB}$. To define $\bar{v}$ start choosing $a \in \mathbf{A}$, $\beta : c \to i(a)$, with the convention $a = a'$, $\beta = 1_{ia}$ if $b = i(a')$ for some $a' \in \mathbf{A}$.

Given $c \to c'$, there exists an arrow (unique, since $\mathbf{C}$ is a groupoid) $\alpha : a \to a'$ such that the diagram

$$\begin{array}{ccc} c & \xrightarrow{\delta} & c' \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ i(a) & \xrightarrow[i(\alpha)]{\quad} & i(a') \end{array} \tag{39}$$

commutes (better to say: we can find $\theta \in \mathbf{C}(i(a), i(a'))$ such that $\beta'\delta = \theta\beta$, and now since i is fully faithful, $\theta = i(\alpha)$ for a unique α). Since p is a fibration, $v(\beta) : v(c) \to p(u(a))$ and $v(\beta') : v(c') \to p(u(a'))$ can be lifted to arrows $\varepsilon : e \to u(a)$ and $\varepsilon' : e' \to u(a')$, in such a way the diagram

$$\begin{array}{ccc} e & \xrightarrow{\quad} & e' \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon'} \\ u(a) & \xrightarrow[u(\alpha)]{\quad} & u(a') \end{array} \tag{40}$$

can be completed by $e \to e'$ which we define to be $\bar{v}(c \to c')$.

Factorization properties. Let $f : \mathbf{G} \to \mathbf{H}$ be any functor between groupoids, and consider the triangle

$$\begin{array}{ccc} \mathbf{G} & \xrightarrow{\quad i \quad} & (\mathbf{H} \downarrow f) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p} & \\ & \mathbf{H} & \end{array} \tag{41}$$

where $(\mathbf{H} \downarrow f)$ is the comma category between the identity functor of $\mathbf{H}$ and f[9]. We are to prove that

- The functor $p : \big(\alpha : h \to f(g)\big) \mapsto h$ is a fibration;
- The functor $i : x \mapsto \big(1_{f(x)} : f(x) \to f(x)\big)$ is an acyclic cofibration;
- $p \circ i = f$ (this is trivial).

Define the functor $r : (\mathbf{H} \downarrow f) \to \mathbf{G} : \big(\alpha : h \to f(g)\big) \mapsto g$; it's easy to see that $ri = 1_{\mathbf{G}}$ and $ir \cong 1_{(\mathbf{H}\downarrow f)}$, so i is an acyclic (=equivalence of categories) cofibration (=injective-on-objects functor): since the categories involved are groupoids any natural transformation between ir and $1_{(\mathbf{H}\downarrow f)}$ suits; so for $\alpha : h \to f(g)$ it suffices to define $\eta_\alpha : ir(\alpha) \Rightarrow \alpha$ to

$$\begin{array}{ccc} fg & \xrightarrow{ir(\alpha)} & fg \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ h & \xrightarrow[\alpha]{\quad} & fg \end{array}$$

[9] Its objects are arrows $\alpha : h \to f(g)$, arrows $\alpha \Rightarrow \alpha'$ are defined by pairs of arrows $h \to h'$, $g \to g'$ such that a suitable square commutes.

be the pair of arrows $u = \alpha^{-1}\colon f(g) \to h$, $v = 1_{f(g)}\colon f(g) \to f(g)$, in such a way that the diagram besides commutes.

If now we are given $\beta\colon h \to h'$, and $h' = p\big(\alpha\colon h \to f(g)\big)$, it suffices to choose $(\beta, 1_g)$ as a morphism $\alpha\beta \nrightarrow \alpha$, to get the desired lifting, so $p \in$ FIB.

Remark 10. Notice that the factorization can be chosen to be *functorial*.

EXERCISE 8 : *Show that in the category* **Sets** *any function $\phi\colon X \to Y$ can be factored as the composition of a surjective function $X \twoheadrightarrow T$, followed by an injective function $T \hookrightarrow Y$.*

To show that (FIB ∩ WK, COF) is again a factorization system, factor the function on object $f\colon \mathbf{G} \to \mathbf{H}$ as the composition

$$\begin{array}{ccc} G_o & \xrightarrow{i_o} & G_o \amalg H_o \\ & {\scriptstyle f_o}\searrow \quad \swarrow{\scriptstyle p_o} & \\ & H_o & \end{array} \tag{42}$$

where i_o is the obvious embedding in the coproduct and $p = (f_o, 1_{H_o})$. Now the correspondence i_o obviously lift to a functor $\mathbf{G} \to \mathbf{G} \amalg \mathbf{H}$ (precisely the embedding into the coproduct); any definition of a functor p (if it exists) will be a surjective-on-objects functors, so it remains only to prove that it is possible to define $p_a\colon G_a \amalg H_a \to H_a$ to be a bijection. □

3 Torsors and stacks in a topos.

For all the rest of the section $\mathcal{E} = \mathrm{Sh}(\mathbf{C}, \mathrm{COV}_{\mathbf{C}})$ is a Grothendieck topos.

Definition 3.1 (Category of G-torsors)**.** Let $G \in \mathrm{Grp}(\mathcal{E})$; a *(right) G-torsor* consists of a G-object $\emptyset \neq E \in \mathcal{E}^G$ such that the right action defines an isomorphism

$$E \times G \xrightarrow{\langle \pi, a \rangle} E \times E. \tag{43}$$

A morphism of G-torsors consists of an arrow $f\colon F \to E$ such that the diagram besides commutes.

$$\begin{array}{ccc} F \times G & \xrightarrow{f \times 1_G} & E \times G \\ {\scriptstyle a_F}\downarrow & & \downarrow{\scriptstyle a_E} \\ F & \xrightarrow[f]{} & E \end{array}$$

Remark 11. The category of G-torsors in $\mathcal{E}$ is a groupoid.

Proof. See **[Johnstone]**, Lemma **8.31.i**. □

Torsors solve the following problem:

> Given $S \in \mathcal{E}$ find all objects T which are *locally isomorphic* to S, i.e. such that there exists a covering $\{K_i \to T\}$ (in the sense of the topos) and an isomorphism $\big(K = \amalg K_i\big) \times T \to K \times S$ in $\mathcal{E}/K$:

$$\begin{array}{ccc} K \times T & \xrightarrow{\cong} & K \times S \\ & \searrow \quad \swarrow & \\ & K. & \end{array} \tag{44}$$

> This amounts to ask that there is an isomorphism in the internal sheaf $\underline{Iso}(S,T)$; so there is an open covering on which there are sections (which by no means implies that this sheaf has a global section!).

Fix $S \in \mathcal{E}$ and T locally isomoprhic to S, then $E = \underline{Iso}(S,T)$ is a (right) $\mathrm{Aut}(S)$-torsor; if we denote $G = \mathrm{Aut}(S)$, then T can be recovered since it is isomorphic to $E \otimes_G S$ via the *evaluation* morphism

$$\begin{aligned} \mathrm{ev}\colon E \otimes_G S &\longrightarrow T \\ [\alpha, s] &\longmapsto \alpha(s) \end{aligned} \tag{45}$$

where $E \otimes_G S := \mathrm{coeq}\Big(E \times G \times S \overset{E\times a_S}{\underset{a_E \times S}{\rightrightarrows}} E \times S \Big)$ and $a_S\colon G \times S \to S$ is the obvious restriction of $\mathrm{ev}\colon S^S \times S \to S$ to $\mathrm{Aut}(S)$. In other words, we are identifying $(a_E(e,g),s)$ and $(e, a_S(g,s))$ for any $e \in E, s \in S, g \in G$, or (it is equivalent) we are taking the orbit-object $(E \times S)/G$ for the action $g.(e,s) = (e.g^{-1}, g.s)$.

The correspondence $T \mapsto \mathrm{Iso}(S,T)$ defines a bijection

$$\left\{ \begin{matrix} \text{objects locally} \\ \text{isomorphic to } S \end{matrix} \right\}_{/\cong} \longrightarrow \left\{ \begin{matrix} G\text{-torsors} \\ \text{in } \mathcal{E} \end{matrix} \right\}_{/\cong} \tag{46}$$

A G-torsor *over* $X \in \mathcal{E}$ is a G-torsor in the topos $\mathcal{E}/X$.

Example 3.1. A G-torsor over $X \in \mathbf{SSet} = [\Delta^{\mathrm{op}}, \mathbf{Sets}]$ is a principal bundle with base X.

Definition 3.2 (Internal groupoid)**.** An *internal groupoid* $\mathbb{G}$ in $\mathcal{E}$ is a category object $(\mathbb{G}, s, t, e, c)$ in $\mathcal{E}$ such that there exists an additional arrow $i\colon C_a \to C_a$, termed *inversion*, which turn the diagrams

$$\begin{array}{ccc} C_a & \xrightarrow{\Delta} & C_a \times C_a \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle c\circ(i\times \mathrm{id})} \\ C_o & \xrightarrow[e]{} & C_a \end{array} \qquad \begin{array}{ccc} C_a & \xrightarrow{\Delta} & C_a \times C_a \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle c\circ(\mathrm{id}\times i)} \\ C_o & \xrightarrow[e]{} & C_a \end{array} \tag{47}$$

into commutative ones.

If $\mathbb{G}$ is an internal groupoid in $\mathcal{E}$, there exists a notion of right $\mathbb{G}$-torsor: it is an object $E \neq \emptyset$ with an action $a\colon E \times_{G_o} G_a \to E$ in $\mathcal{E}/G_o$ such that

$$E \times_{G_o} G_a \xrightarrow{\langle \pi, a \rangle} E \times_{G_o} E \tag{48}$$

is an isomorphism. A $\mathbb{G}$-torsor over $X \in \mathcal{E}$ is a $\mathbb{G}$-torsor in $\mathcal{E}/X$.

Isomorphism classes of (right) $\mathbb{G}$-torsors over X are collected in the set $H^1(X, \mathbb{G})$, and this defines a bifunctor

$$H^1\colon \mathcal{E}^{\mathrm{op}} \times \mathbf{Gpd}(\mathcal{E}) \to \mathbf{Sets} \tag{49}$$

For the sake of simplicity, let's treat the two components of H^1 separately:

- Given $g\colon Y \to X$, then g induces a(n essential) geometric morphism $\mathcal{E}/X \to \mathcal{E}/Y$, whose left part $g^*\colon \mathcal{E}/X \to \mathcal{E}/Y$ sends monoid/group objects in monoid/group objects, and G-objects in $\mathcal{E}/X$ in g^*G-objects in $\mathcal{E}/Y$. In practice $g^*(E \times_X G) \to g^*E$ is the G-torsor on Y given by $g^*E \times_Y g^*G \to g^*E$. Something similar happens when we consider *groupoid* actions.

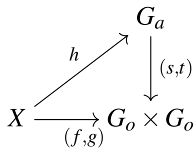
- Given a(n internal) functor $\mathbb{G} \to \mathbb{H}$, notice that $\mathbb{H}$ is both a right and a left $\mathbb{G}$-object, in the obvious way, hence $E \otimes_{\mathbb{G}} \mathbb{H}$ is a right $\mathbb{H}$-torsor over X (see **[Moerdijk], VII.3.4**).

3.1 The category $\hom(X, \mathbb{G})$.

Let $\mathbb{G}$ be a groupoid in $\mathcal{E}$. Define the category $\hom(X, \mathbb{G})$ in such a way that

- its objects are arrows $f \in \mathcal{E}(X, G_o)$;

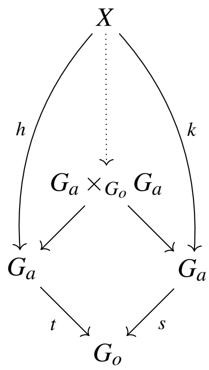
- arrows between $f, g\colon X \to G_o$ are arrows $h\colon X \to G_a$ in $\mathcal{E}$ such that the triangle



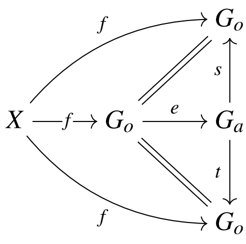
(50)

  commutes.

- Composition of arrows $h\colon f \to g$ and $k\colon g \to m$ is defined once we noticed that $th = sk$, hence $(h,k) \in \mathrm{src}(c)$: we can define $k \diamond h := c \circ (h,k)$, where (h,k) is the unique arrow $X \to G_a \times_{G_o} G_a$ such that the diagram besides commutes (UMP of the pullback).



- The identity arrow $f \to f$ is $e \circ f$, where $e\colon G_o \to G_a$ is the identity of the groupoid. The diagram

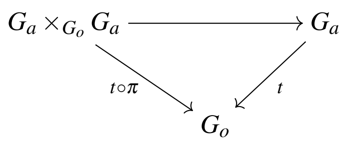


(51)

  commutes everywhere.

- Every $h\colon f \to g$ has an inverse $h^{-1}\colon g \to f$ defined by $i \circ h$, where $i\colon G_a \to G_a$ is the inversion map of $\mathbb{G}$: notice that $s \circ j = t, t \circ j = s$.

Remark 12. The arrow $t\colon G_a \to G_o$ defines a right $\mathbb{G}$-torsor. Indeed, the diagram



(52)

commutes when we define $G_a \times_{G_o} G_a \to G_a$ to be the composition arrow. If $E = G_a$, the arrow $E \times_{G_o} G_a \to E \times E$ is an isomorphism with inverse (using generalized elements $x\colon X \to E$, $y\colon Y \to G_a$) $(x,y) \mapsto (x, c(y, jx))$. In the same manner, $s\colon G_a \to G_o$ defines a left $\mathbb{G}$-torsor.

We can define a functor

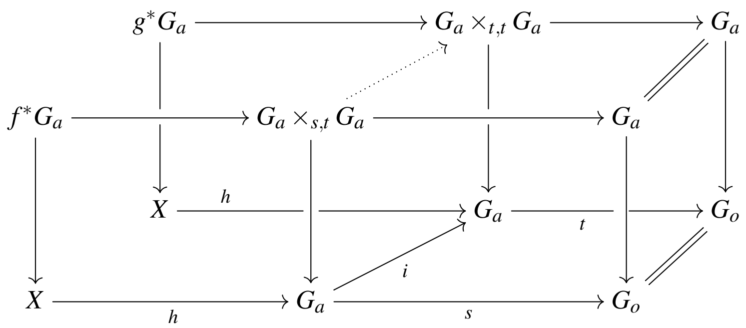
$$F_X\colon \hom(X,\mathbb{G}) \longrightarrow H(X,\mathbb{G}) \tag{53}$$

(where $H(X,\mathbb{G})$ denotes the collection of $\mathbb{G}$-torsors over X: $H^1(X,\mathbb{G})$ consists of the connected components of the groupoid of $\mathbb{G}$-torsors: $\Pi_0\big(H(X,\mathbb{G})\big) = H^1(X,\mathbb{G})$) defined on objects by

$$\big(f\colon X \to G_o\big) \mapsto f^*G_a \tag{54}$$

where f^*G_a is the pullback of $t\colon G_a \to G_o$ (which is a torsor), along $f\colon X \to G_o$.

Consider the diagram



(55)

and for the sake of simplicity consider the case where $\mathcal{E} = \mathbf{Sets}$: then the map $f^*G_a \to g^*G_a$ is defined once we noticed that

$$\begin{aligned} f^*G_a &= \{(x,\alpha) \in X \times G_1 \mid f(x) = t(\alpha)\},\\ g^*G_a &= \{(y,\beta) \in X \times G_1 \mid g(y) = t(\beta)\}. \end{aligned}$$

Given $h\colon X \to G_a$ which sends $x \in X$ to an arrow $f(x) \to g(x)$, having

$$\# \xrightarrow{\ \alpha\ } f(x) \tag{56}$$

we can compose $h(x)$ on the right to obtain

$$\# \xrightarrow{\ \alpha\ } f(x) \xrightarrow{\ h(x)\ } g(x) \tag{57}$$

i.e. the element $(x, h(x)\circ\alpha) \in g^*G_a$.

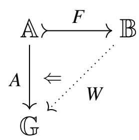
Definition 3.3. The functor F_X can be shown to be fully faithful. The groupoid $\mathbb{G}$ is said to be a *stack* if F_X is an equivalence of categories, for any $X \in \mathcal{E}$ (i.e. if any F_X is essentially surjective: for any $\mathbb{G}$-torsor $E \to X$ there is $f\colon X \to G_o$ such that $E \cong f^*G_a$).

[Joyal-Tierney2] offers a characterization of those groupoids which are stacks in terms of a weak lifting condition which will be shown to distinguish some stacks as the class of fibrant objects for the folk model structure on **Gpd**$(\mathcal{E})$ (which can be easily internalized repeating the proof of Theorem **2.1**): let's first of all show [Joyal-Tierney2]'s characterization of stacks in $\mathcal{E}$.

Theorem 3.1. The following conditions are equivalent for a groupoid $\mathbb{G} \in$ **Gpd**$(\mathcal{E})$.

i. $\mathbb{G}$ is a stack;

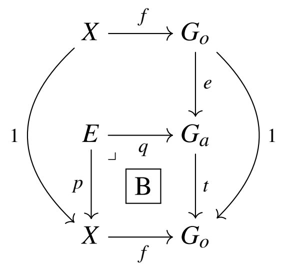
ii. Any $\mathbb{G}$-torsor $E \to X$ admits a section $X \to E$;



iii. Any span of groupoids $\mathbb{G} \leftarrow \mathbb{A} \xrightarrow{F} \mathbb{B}$, where F is an equivalence of categories injective on objects, admits an extension $\mathbb{B} \to \mathbb{G}$ making the triangle besides commutative up to an invertible 2-cell $W \circ F \Rightarrow A$.

iv. Any span of groupoids $\mathbb{G} \leftarrow \mathbb{A} \xrightarrow{F} \mathbb{B}$, where F is an equivalence of categories, admits an extension $\mathbb{B} \to \mathbb{G}$ making the same triangle commutative up to an invertible 2-cell $W \circ F \Rightarrow A$.

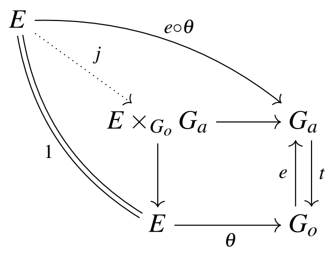
Proof. It's clear that (iv) implies (iii), and it's easy to see that (i) is equivalent to (ii): indeed if $\mathbb{G}$ is a stack, i.e. if any $\mathbb{G}$-torsor $E \to X$ results as the pullback of $t\colon G_a \to G_o$ via some $f_E\colon X \to G_o$, then we can find a section for $E \to X$ thanks to the fact that t admits a section $e\colon G_o \to G_a$ (the arrow of identity, cfr. Definition **1.14**): the diagram



(58)

commutes, and by the UMP of the pullback B, there exists a (unique) $r\colon X \to E$ such that $ef = qr$ and $pr = 1_X$. On the contrary, if any $\mathbb{G}$-torsor admits a section, let $p\colon E \to X$ and $s\colon X \to E$ be such a torsor and such a section. Then the pullback of t along $\alpha \circ s$ is exactly $p\colon E \to X$.

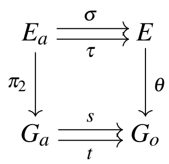
Let's show now that (iv) implies (i): let $E \to X$ be a $\mathbb{G}$-torsor over X, implicitly regarded as an object in $\mathcal{E}/G_o$ (i.e. as an arrow $\theta\colon E \to G_o$) and endowed with an action $\alpha\colon E \times_{G_o} G_a \to E$ (regarded again in $\mathcal{E}/G_o$). Then we can define a groupoid $\mathbb{E}$ having objects $E_o = E$ and arrows $E_a = E \times_{G_o} G_a$, identity $j\colon E \to E_a$ determined as the unique arrow completing the diagram



(59)

(use the UMP of the pullback!) and having as source-target pair (σ, τ) the arrows $(\alpha, \pi_1)\colon E \times_{G_o} G_a \to E \times E$. Composition of arrows is the law $\diamond\colon E_a \times_{E_o} E_a \to E_a$ defined by $(u, f) \diamond (v, g) = (u, f.g)$, for any pair $(u, f), (v, g) \in E_a = E \times_{G_o} G_a$ (the dot is the composition in $\mathbb{G}$): axioms (**25**)-(**28**) for an internal category are easily seen to hold.

Now, the correspondence



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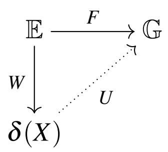
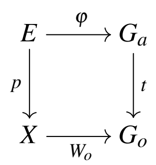


defines a(n internal) functor $F\colon \mathbb{E} \to \mathbb{G}$. If now we call $\delta(X)$ the discrete category on an object X (which is simply obtained by putting $(\delta X)_o = (\delta X)_a = X$ and $s = t = e = 1_X$; the composition is again 1_X, once we identified $X \times_{s,t} X \cong X$), then there is an equivalence of categories[10] $W\colon \mathbb{E} \to \delta(X)$, which by (iv) admits a filler $W\colon \delta(X) \to \mathbb{G}$ in the diagram besides.

This triangle commutes up to an invertible 2-cell $\varphi\colon F \Rightarrow UW$; in other words there is an arrow $\varphi\colon E \to G_a$: the naturality condition for φ exactly amounts to ask that the square



(61)

commutes and is a pullback, so the groupoid $\mathbb{G}$ is a stack as $f^* G_a \cong E$. □

Definition 3.4. Let $\mathbb{G} \in \mathbf{Gpd}(\mathcal{E})$; we define a *stack completion* for $\mathbb{G}$ an equivalence of categories $\mathbb{G} \to \overline{\mathbb{G}}$, where $\overline{\mathbb{G}}$ is a stack.

[10] see [**Bunge**] and [**Everaert**] for the definition: the rough idea is that an internal equivalence is a pair of functors with two natural transformations $\eta\colon 1 \Rightarrow G \circ F$ and $\varepsilon\colon F \circ G \Rightarrow 1$ which are "componentwise isomorphisms". Nevertheless one has to pay attetion to some subtleties (which we do not intend to unravel) linked to the *internal logic* of the topos, namely the fact that in a generic topos the Axiom of Choice, which is equivalent to the statement "any fully faithful, essentially surjective functor lifts to an equivalence of categories" usually fails to hold. The fussy reader might distinguish between the two notion of *weak* equivalence of categories (=fully faithfulness and essential surjectivity) or *strong* equivalence (=having a quasi-inverse).

Remark 13. Any two stack completions of $\mathbb{G}$ are equivalent as groupoids. If $\mathbb{G}$ admits a stack completion, functoriality of $H(X,-)$ entails that $H(X,\mathbb{G}) \cong H(X,\overline{\mathbb{G}}) \cong \hom(X,\overline{\mathbb{G}})$, hence $\overline{\mathbb{G}}$ *represents* the functor $H(-,\mathbb{G})$.

3.2 Strong stacks as fibrant objects.

Following [Joyal-Tierney2], a *strong stack* is a groupoid $\mathbb{G} \in \mathbf{Gpd}(\mathcal{E})$ such that condition (*iii*) in Theorem **3.1** holds on the nose, i.e. if any span of groupoids $\mathbb{G} \leftarrow \mathbb{A} \xrightarrow{F} \mathbb{B}$, where F is an acyclic cofibration in the internal folk model structure, admits an extension $W : \mathbb{B} \to \mathbb{G}$ such that $W \circ F = A$.

The notion of strong stack completion is the exact analogue of Definition **3.4**.

Definition 3.5. Let $\mathbb{G} \in \mathbf{Gpd}(\mathcal{E})$; we define a *strong stack completion* for $\mathbb{G}$ an equivalence of categories $\mathbb{G} \to \overline{\mathbb{G}}$, where $\overline{\mathbb{G}}$ is a strong stack.

Remark 14. Any two strong stack completions of $\mathbb{G}$ are equivalent as groupoids. We can prove that any $\mathbb{G} \in \mathbf{Gpd}(\mathcal{E})$ admits a strong stack completion: indeed strong stacks can be characterized as *fibrant objects* for the folk model structure on $\mathbf{Gpd}(\mathcal{E})$; hence we can see $\mathbb{G} \to \overline{\mathbb{G}}$ to be the object part of the *fibrant replacement* functor.

Let's make the discussion more precise:

Theorem 3.2 (Folk model structure on $\mathbf{Gpd}(\mathcal{E})$)**.** The category $\mathbf{Gpd}(\mathcal{E})$ of internal groupoids in $\mathcal{E}$ admits a model structure such that

- $\text{W}\textsc{k}$ is the class of (internal) equivalences of groupoids;
- Cofibrations are internal functors which are injective on objects;
- Fibrations are functors in $\textsc{rlp}(\text{W}\textsc{k} \cap \text{C}\textsc{of})$.

The proof consists in rephrasing Theorem **2.1** in the internal semantics of the topos $\mathcal{E}$. See also [Joyal-Tierney2].

In the internal folk model structure there is an analogous characterization of acyclic fibrations in terms of anodyne maps:

Lemma 3.1. An arrow $p : \mathbb{E} \to \mathbb{B}$ between internal groupoids in $\mathcal{E}$ is an anodyne fibration (i.e., it lifts any cofibrations) if and only if it is an internal equivalence of categories and $E_o \to B_o$ is injective as an object of $\mathcal{E}/B_o$[11].

Proof. Given the adjunction $\delta \dashv (-)_0$ established in [Gray] (which can be easily internalized in $\mathcal{E}$), any diagram like the following left one

$$\begin{array}{ccc} A & \longrightarrow & E_o \\ \downarrow & \nearrow & \downarrow{\scriptstyle p_o} \\ C & \longrightarrow & B_o \end{array} \qquad \begin{array}{ccc} \delta A & \longrightarrow & \mathbb{E} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \delta C & \longrightarrow & \mathbb{B} \end{array} \tag{62}$$

[11] It is not so surprising that the $\textsc{rlp}$ is somewhat linked to a lifting condition: try to describe what "being an injective object" mean for an object $A \to X$ of $\mathbf{C}/X$.

is equivalent to a diagram like one on the right in $\mathbf{Gpd}(\mathcal{E})$. The second one admits a dotted filler, since the arrow $\delta A \to \delta C$ is a cofibration. On the other hand, using again the same adjunction, this implies that the initial square admits a diagonal filler $C \to E_o$, hence p is an injective object in $\mathcal{E}/B_o$. The fact that anodyne fibrations are equivalences follows immediately rewriting in $\mathcal{E}$ the proof given for **Sets** during Theorem **2.1**.

Conversely, suppose $p\colon \mathbb{E} \to \mathbb{B}$ is fully faithful and p_o is an injective object in $\mathcal{E}/B_o$; given a cofibration $\mathbb{A} \hookrightarrow \mathbb{C}$ fitting in the square

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{E} \\ \downarrow & & \downarrow p \\ \mathbb{C} & \longrightarrow & \mathbb{B} \end{array} \tag{63}$$

then there exists a lifting once we apply the functor $(-)_0$, since p is injective as an object of $\mathcal{E}/B_o$. This is the object-part of the desired lifting $g\colon \mathbb{C} \to \mathbb{E}$. We can now exploit fully faithfulness to define g on arrows (and for the sake of clarity we use generalized elements in the topos): given $\alpha \in C_a$, $f(\alpha) = pg(\alpha)$, which comes from a unique arrow $\beta := g(\alpha)$. □

In the case $\mathcal{E} = \mathbf{Sets}$ this is precisely the definition of anodyne fibration.

$$\begin{array}{ccc} E_a & \xrightarrow{t} & E_o \\ {\scriptstyle f_a}\downarrow & & \downarrow{\scriptstyle f_o} \\ B_a & \xrightarrow[t]{} & B_o \end{array}$$

Definition 3.6. A functor $f\colon \mathbb{E} \to \mathbb{B}$ between groupoids in $\mathcal{E}$ is said to be *discrete fibration* if the square besides is a pullback.

As the name may suggest, discrete fibrations are particular fibrations for the internal folk model structure: they are precisely those fibration which lift *uniquely* any acyclic cofibration.

This characterization becomes evident when we notice that in the case $\mathcal{E} = \mathbf{Sets}$ a discrete fibration between small groupoids is simply a functor such that the lifting property defining fibrations holds *uniquely*: being a discrete fibration means that the natural map of sets

$$\begin{array}{rcl} C_a & \longrightarrow & \{(\alpha, C) \mid f(c) = s(\alpha)\} \\ \beta & \longmapsto & \binom{F(\beta)}{s(\beta)} \end{array} \tag{64}$$

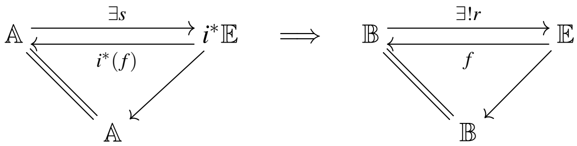
is a bijection.

Proof. First of all let's recall that (an internalization of) a general result in Category Theory asserts an equivalence of categories between $\mathbb{B}$ based discrete fibrations and (internal) presheaves on $\mathbb{B}$. Let's limit ourselves to the case $\mathcal{E} = \mathbf{Sets}$: then this equivalence can be realized seding a $\mathbb{B}$-presheaf F on the forgetful functor $\int F \to \mathbb{B}$, which is easily seen to be a discrete fibrations[12], and a discrete fibration $f\colon \mathbb{E} \to \mathbb{B}$ in the presheaf $B \mapsto \{E \mid f(E) = B\}$.

Since discrete fibrations are stable under pullback (as it can be immediately seen), it suffices to show that if $i\colon \mathbb{A} \to \mathbb{B}$ is an acyclic cofibration in $\mathbf{Gpd}(\mathcal{E})$, and if $i^*(\mathbb{E})$ has a

[12] The category $\int F$ is the *category of elements* of F, defined having as objects pairs (b, x), where $B \in \mathbb{B}$ and $x \in F(B)$; the general definition uses internal functor as defined in **[Borceux]**, §**8.2**: a $\mathbb{B}$-based presheaf consist of a suitably defined functor $\mathbb{B}^{\mathrm{op}} \to \mathcal{E}$.

section s over $\mathbb{A}$, then there exists a *unique* section r of f over $\mathbb{B}$, extending s. In other words we have to prove that



(65)

This is indeed true, as i^* is an equivalence of categories, and a section s of $i^*(f)$ over $\mathbb{A}$ is an arrow $1_{\mathbb{A}} \to i^*(f)$: since i^* is full and faithful, and $i^*(1_{\mathbb{B}}) \cong 1_{\mathbb{A}}$ there exists a *unique* $r\colon 1_{\mathbb{B}} \to \mathbb{E}$ over $\mathbb{B}$ such that $i^*(r) = s$, and such r has exactly the desired property. □

Remark 15. The preceding result provides a number of examples of strong stacks: In fact, if we know that $\mathbb{E} \to \mathbb{G}$ is a discrete fibration and $\mathbb{G}$ is a strong stack, then $\mathbb{E}$ is a strong stack too: this follows from the fact that FIB is a compositive class in any model structure, hence $\mathbb{E} \twoheadrightarrow \mathbb{G} \twoheadrightarrow *$ must be a fibration, hence

> the category of elements of a $\mathbb{G}$-torsor $\mathbb{E}$ is always a strong stack

(a result which is, more or less tacitly, used in **[Joyal-Tierney2]**). Again, for any object $X \in \mathcal{E}$ the discrete category δX is a strong stack with unique lifting, since $\delta X \to *$ is a discrete fibration ($*$ is the discrete category on the terminal object, i.e. the terminal object of Cat$(\mathcal{E})$: for a comprehensive treatment of limits and colimits internal to a complete/cocomplete category cf. **[Borceux]**, §**8**).

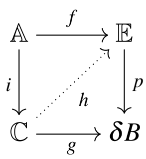
There are indeed other equivalent conditions to recognize whether a given groupoid $\mathbb{G}$ is a strong stack:

Proposition 3.1. Let $p\colon E \twoheadrightarrow B$ an epimorphism in $\mathcal{E}$, and let $\mathbb{E}$ be the equivalence relation obtained by p pulling it back along itself; one can see it as a groupoid having objects E and source-target pair $(s,t)\colon E \times_B E \hookrightarrow E \times E$, and it is possible to prove that this groupoid is exactly the groupoid of elements of the torsor over B, as it has been defined along the proof of Theorem **3.1**; then $\mathbb{E}$ (regarded as an internal groupoid) is a strong stack if and only if p is ain injective object in $\mathcal{E}/B$.

Proof. The characterization of acyclic fibrations implies that the equivalence of categories $\mathbb{E} \to \delta B$ induced by p is an acyclic fibration if and only if its object part $p\colon E \to B$ is an injective object in $\mathcal{E}/B$, i.e. if and only if p lifts any cofibration.

The discrete groupoid δB is always a strong stacks, hence if we suppose that $p\colon E \to B$ is injective, then $\mathbb{E}$ is a strong stack too.

Conversely, suppose $\mathbb{E}$ is a strong stack and $i\colon \mathbb{A} \hookrightarrow \mathbb{C}$ is an acyclic cofibration, in the square



(66)

we can find a dotted filler such that at least the upper square commutes. The lower square now leaves us with two functors $ph, g\colon \mathbb{C} \to \delta B$ which coincide on $\mathbb{A}$, hence on $\mathbb{C}$ because $\mathbb{E} \to \delta B$ is a discrete fibration. $\square$

Proposition 3.2. An internal groupoid $\mathbb{G}$ in $\mathcal{E}$ is a strong stack if and only if for any $X \in \mathcal{E}$, any $\mathbb{G}$-torsor $E \to X$ is an injective object in $\mathcal{E}/X$, namely if for any monomorphism $A \hookrightarrow B$, any commutative square

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & X \end{array} \tag{67}$$

admits a dotted filler.

To conclude the discussion, we present an example of a strong stack completion, as it is given in **[Joyal-Tierney2]**; recall (e.g. from **[Johnstone]**, §**8**) that the category of internal abelian groups in a topos is an abelian category with enough injectives, monadic over $\mathcal{E}$ (in particular the forgetful functor $U\colon \mathrm{Ab}(\mathcal{E}) \to \mathcal{E}$ admits a left adjoint, the *free internal abelian group functor* $X \mapsto \mathbb{Z}^{(X)}$).

Hence given an internal abelian group $A \in \mathrm{Ab}(\mathcal{E})$ we can embed it into an injective object J and we can consider the quotient $B = J/A$. If we write $p\colon J \to B$ for the natural quotient map, and let $(s,t) = (\pi_B, q)$ be arrows $B \times J \to B \times B$, defined respectively as the first projection and the arrow $(x, j) \mapsto x + p(j)$, then we have the following

Proposition 3.3. The arrangement of objects and arrows $(s,t)\colon B \times J \to B \times B$ as defined before is a groupoid $\overline{\mathbb{A}}$ where composition is given by addition, and it is a strong stack completion of A (regarded as a one-object groupoid $\mathbb{A} = (A_o, A_a) = (*, A)$ where $s,t\colon A \to *$, and composition is the internal addition).

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