

(not so) Categorical tools II

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As someone said, “Mathematical Analysis is a huge variation on the single theme of continuity”. In the same vein, there’s no other problem in Homotopy Theory than the computation of homotopy groups of spheres. In his paper [Serre], inspired by the work of Leray [Leray], Serre built a spectral sequence which expresses the homology of the total space in a fibration in terms of the homology of the base, taking values in the homology of the fiber, and exploited it to produce the first nontrivial computation of the homotopy group of a sphere:

$$\pi_4(S^3) \cong \mathbb{Z}/2 \quad \pi_5(S^3) \cong \mathbb{Z}/2.$$

Several highly nontrivial computations stemmed from this initial spark of genius:

- the (co)homology ring of classifying spaces;
- the (co)homology of classical Lie groups;
- the (co)homology of Grassmannian manifolds;
- a derivation of characteristic classes of a bundle...

He who wants to master the witchcraft of topological machineries must start grounded on earth, starting from examples and explicit computations, and then elevating himself layer after layer.

The poor in spirit could one day realize that everything computable in Algebraic Topology, is computable via a spectral sequence:

- homotopy groups of spectra/sphere spectrum [Adams];
- generalized cohomology theories [Atiyah-Hirzebruch];
- homotopy (co)limits of functors [Bousfield-Kan], [Thomason];
- Group cohomology [Lyndon-Hochschild-Serre];
- Lie algebra cohomology [Van Est]...

References

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