Street-Walters-Yoneda structures

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What is FCT

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Yoneda Structures on 2-Categories

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What is FCT

Formal category theory is a branch of 2-category theory.

It serves to axiomatize the structural part of category theory.

It starts from noticing that there are many category theories:

- **Ordinary** category theory.
- **Enriched** category theory.
- **Internal** category theory.
- **Fibred** and indexed category theory.
- **(Symmetric) monoidal** (braided, traced,...) category theory.

What is FCT

In each flavour of category theory, we have essentially the **same definitions** and theorems.

- Presheaves and the Yoneda lemma.
- Adjoint functor theorems.
- Monadicity theorems.
- Presentability and duality.

What is FCT

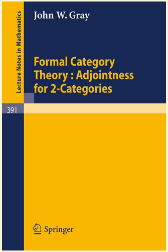
As category theorists, this situation calls to us for abstraction.

If we see essentially the same theorem being reproven again and again in different settings, we should hope that each variant is a consequence of a more general statement.

This is the motivation for formal category theory.

> *Formal category theory is the application of the philosophy of category theory to category theory.*

What is FCT



Category theory is a theory we can interpret in different contexts. These contexts are 2-categories.

Our task as category theorists is to unravel the properties enabling to treat an abstract 2-category $\mathcal{K}$ as if it were **Cat**.»

This is not a new idea:

- **topos theory**: treat $\mathcal{K}$ as if it were **Set**;
- **categorical algebra**: treat $\mathcal{K}$ as if it were **Alg**(T);
- **homological algebra**: treat $\mathcal{K}$ as if it were **Ch**(R);
- formal category theory: treat $\mathcal{K}$ as if it were **Cat**.

What is FCT

Perhaps surprisingly, the bare structure of a 2-category does not suffice to embody 'all' category theory in a formal way.

Have:

- **co/limits**
- **adjunctions**: pairs of 1-cells $f : X \leftrightarrows Y : g$
- **monads**: endo-1-cells $t : A \to A$
- **Kan extensions** w/ their universal properties
- **fibrations** as algebras of a monad

Don't have:

- a pointwise formula to compute Kan extensions
- homset characterizations of adjunctions: $Y(f, 1) \cong X(1, g)$
- Yoneda lemma
- calculus of modules / profunctors

What is FCT

To unravel the structural part of category theory we have to elucidate its essential nature. So, we first have to answer the following **simple question**:

> *What is category theory?*

With time, different answers were proposed. There's a tension between two complementary, but not independent, approaches:

- (**Y**) Category theory is the class of **corollaries of Yoneda lemma**.
- (**E**) Category theory is a **representation theory on steroids.**

What is FCT

Following (**Y**) leads to the theory of *Yoneda structures*.

- Street, Ross, and Robert Walters. ”Yoneda structures on 2-categories.” Journal of Algebra 50.2 (1978): 350-379.
- Weber, Mark. ”Yoneda structures from 2-toposes.” Applied Categorical Structures 15 (2007): 259-323.
- Walker, Charles. ”Yoneda structures and KZ doctrines.” Journal of Pure and Applied Algebra 222.6 (2018): 1375-1387.

Following (**E**) leads to the theory of **proarrow equipments**.

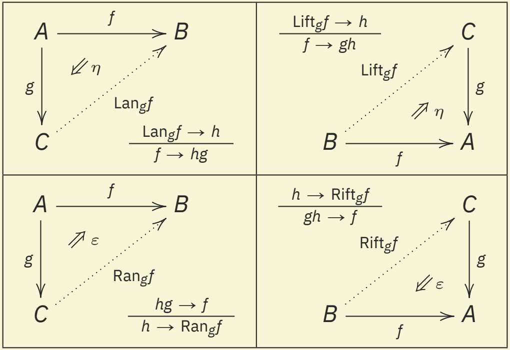
- Wood, Richard J. ”Abstract pro arrows I.” Cahiers de topologie et géométrie différentielle 23.3 (1982): 279-290.
- Rosebrugh, Robert, and R. J. Wood. ”Proarrows and cofibrations.” Journal of Pure and Applied Algebra 53.3 (1988): 271-296.
- Cruttwell, Geoffrey SH, and Michael A. Shulman. ”A unified framework for generalized multicategories.” arXiv:0907.2460 (2009).

Preliminaries

Introduction

Fundamental to our discussion are some basic notions of 2-category theory.

First of all, the notion of a Kan arrow.



pointwise... ...absolute

Introduction

Adjoints are everywhere, but everything is a Kan extension (Mac Lane, 1972)

...(or a Kan lift). (Street and Walters, 1978)

- Colimits are Kan extensions
- Weighted limits are Kan extensions
- adjoints are Kan extensions / lifts

Theorem (formal characterization of adjoints)

The following are equivalent to $F \dashv G$:

- F is an absolute right extension of 1 along G;
- G is an absolute left extension of 1 along F;
- F is an absolute left lift of 1 along G;
- G is an absolute right lift of 1 along F.

Yoneda structures

Axioms

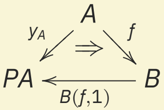
Let $\mathcal{K}$ be a 2-category. A yoneda structure in the sense of Street and Walters on $\mathcal{K}$ specifies:

- A class of 1-cells called *tight* arrows (SW terminology: admissible 1-cells); an object is then called tight if its identity 1-cell is tight;

  $\rightsquigarrow$ the locally small categories

- A choice of a tight 1-cell $y_A : A \to PA$ for each tight object A;

  $\rightsquigarrow$ the Yoneda embeddings

- A choice of extensions $\langle B(f,1), \chi^f \rangle$: 

for each tight 1-cell $f\colon A \to B$ with tight domain A.

  $\rightsquigarrow$ the functor $B(f,1) : b \mapsto (a \mapsto B(fa,b))$

Axioms

These data are subject to the following axioms:

Axiom 1

The extension $\langle B(f,1), \chi^f \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_f\, y_A$.

In $\mathcal{K} = \mathbf{Cat}$ this is true because $\mathrm{Lan}_f y_A$ can be computed pointwise, as the colimit

$$b \mapsto \mathrm{colim}\left(f/b \longrightarrow A \longrightarrow [A^o, \mathbf{Set}] \right)$$

which coincides precisely with the functor $B(f, b)$.

Axioms

Axiom 2

The pair $\langle f, \chi^f \rangle$ exhibits the absolute left lifting $\mathrm{Lift}_{B(f,1)} y_A$.

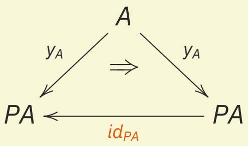
In $\mathcal{K} = \mathbf{Cat}$ this is true because the pair $\langle f, \chi^f \rangle$ exhibits a relative adjunction $f \;{}_{y_A}\!\dashv B(f, 1) = N_f$.

$$
\begin{aligned}
\mathbf{Cat}(A, PA)\big(y_A, N_f \cdot g\big) &\cong \int_x [A^o, \mathbf{Set}]\big(y_A x, N_f \cdot g(x)\big) && \text{(def)} \\
&\cong \int_x [A^o, \mathbf{Set}]\big(y_A x, B(f-, gx)\big) && \text{(def)} \\
&\cong \int_x B(fx, gx) && \text{(yl)} \\
&\cong \mathbf{Cat}(A, B)(f, g)
\end{aligned}
$$

Axioms

Axiom 3

The pair $\langle id_{PA}, id_{y_A} \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_{y_A} y_A$



In $\mathcal{K} = \mathbf{Cat}$ this is true because

- ▷ every presheaf is a colimit of representables

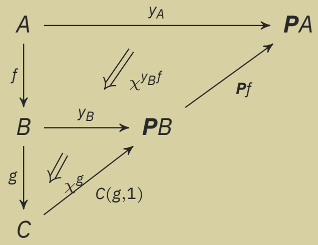
One uses the pointwise formula for $\mathrm{Lan}_{y_A} y_A$ to see that this is equivalent to the fact that

- ▷ the Yoneda embedding is a dense functor.

Axioms

Axiom 4

Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells



exhibits the pointwise extension $\mathrm{Lan}_{g \cdot f}\, y_A$.

Axioms

Corollary

P is a pseudofunctor $T(\mathcal{K})^{coop} \longrightarrow \mathcal{K}$ with domain the 1-2-full subcategory of tight objects of $\mathcal{K}$.

Proof.

Given $f : A \to B$, define $Pf := PB(y_B \cdot f, 1)$. **Axioms 3-4** give invertible comparators

$$id_{PX} \Longrightarrow PX(id_X \cdot y_X, 1) = PX(y_X, 1)$$
$$Pf \cdot Pg \Longrightarrow P(g \cdot f)$$

through the universal property of the extensions involved. Having pointwise and absolute extensions is fundamental. Pseudofunctoriality follows from very boring diagram chasing. □

On the notation $B(f, 1)$

Define $B(f,f) := B(f,1) \cdot f$; then

$$\chi^f : y_A \Longrightarrow B(f,f)$$

plays the same role of the natural transformation $A(a,a') \to B(fa,fa')$ and in fact

Theorem (pointwise characterization of adjoints)

If $f : A \leftrightarrows B : g$ is an adjoint pair, then we have an isomorphism of 1-cells

$$B(f,1) \cong A(1,g)$$

Category theory in a Yoneda structure

Definition (weighted colimit in FCT)

Given tight $A, f : A \to B$ and $M, j : M \to PA$ the j-weighted colimit for f, written $j \otimes f : M \to B$, is the j-relative left adjoint of $B(f, 1)$. This means that

$$B(j \otimes f, 1) \cong PA(j, B(f, 1))$$

In **Cat**, given $f : A \to B$, $j : M \to PA$ (often only given when $M \cong 1$)

$$B(j \otimes f, 1) \cong [A^o, Set](j, B(f, 1))$$

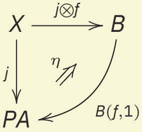
Category theory in a Yoneda structure

Theorem: LAPC

A left adjoint 1-cell $l : A \to B$ preserves all j-indexed colimits that exist in $\mathcal{K}$ and that can be composed with l.

Proof.

Assume $l \overset{\eta}{\dashv} r$ and the lifting



exhibiting $j \otimes f$ is given; then

$$\begin{aligned} PA(j, X(l \cdot f, 1)) &\cong PA(j, B(f, r)) = PA(j, B(f, 1)) \cdot r \\ &\cong B(j \otimes f, 1) \cdot r = B(j \otimes f, r) \\ &\cong X(l \cdot (j \otimes f), 1) \end{aligned}$$

thus exhibiting $j \otimes (l \cdot f)$. $\square$

To wrap up

- we know how to do category theory in **Cat**;
- we want to perform similar computations in a generic 2-category;
- definitions/notation are engineered to suggest the analogy with the classical case;
- proofs proceed formally, with a calculus;
- formal category theory is the endeavour of writing down proofs that work not only in **Cat**, but in any Yoneda structure;
- formal category theory might be thought as the internal language of a 2-category;

Weber, Mark. "Yoneda structures from 2-toposes.".

Guitart, René. "Qu'est-ce que la logique dans une catégorie ?".

Conclusion

Why do I care about this?

- it's an elegant way to do CT, without worrying about the trifling implementation details of universal objects;
- FCT has for CT the same unification power that CT has for abstract algebra and logic;

$$FCT : CT = CT : Math$$

...so perhaps $FCT = \frac{CT^2}{Math}$

- the arguments are formal derivations that can be, in principle, taught to a computer.

Conclusion

Street and Walters' work has been a source of inspiration for many other authors, including me:

- Di Liberti, Ivan, and —. "Accessibility and presentability in 2-categories." Journal of Pure and Applied Algebra 227.1 (2023): 107155.
- Di Liberti, Ivan, and —. "On the unicity of formal category theories." arXiv:1901.01594 (2019).
- Arkor, Nathanael, Ivan Di Liberti, and —. "Adjoint functor theorems for lax-idempotent pseudomonads." arXiv:2306.10389 (2023).
- ...more to come!