Bicategories for automata theory

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Line of research: a series j/w G. Boccali, A. Laretto, S. Luneia; EPTCS.397.1 .

Actually, there is more to this story:

- Boccali, G., Laretto, A., ____, & Luneia, S. *Completeness for categories of generalized automata*. LIPIcs.CALCO.2023.20 ;
- Boccali, G., Femić, B., Laretto, A., ____, & Luneia, S. *The semibicategory of Moore automata*. arXiv:2305.00272 .
- ____, *Automata and coalgebras in categories of species*, arXiv.2401.04242 , Proceedings of CMCS.

Fix an ambient monoidal category $\mathcal{K}$.

Classically (cf. Ehrig *et al*) one studies the category $Mly(A, B)$ having

- objects the spans $X \overset{d}{\leftarrow} A \otimes X \overset{s}{\rightarrow} B$;
- morphisms the $f : X \to Y$ 'compatible with d and s' in the obvious sense:

$$\begin{array}{ccccc} X & \longleftarrow & A \otimes X & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle A \otimes f} & & \| \\ Y & \longleftarrow & A \otimes Y & \longrightarrow & B \end{array}$$

and the category $Mre(A, B)$ having objects the 'disconnected' spans $X \leftarrow A \otimes X, X \to B$ and a similar choice of morphisms.

The results in this direction are essentially three:

- if $T : \mathcal{K} \to \mathcal{K}$ is a commutative monad, Mealy and Moore machines in the (monoidal) Kleisli category $\mathcal{K}_T$ are ‘non-deterministic’ machines for a notion of fuzziness fixed by T;
- if $\mathcal{K}$ is closed, one can characterize Mealy and Moore machines coalgebraically [Jacobs, 2006], and in particular provide a slick proof of the co/completeness of $Mly(A, B)$ and $Mre(A, B)$;
- if $\mathcal{K}$ is Cartesian monoidal, $Mly(A, B)$ is the hom-category of a bicategory **Mly**, and $Mre(A, B)$ the hom-category of a semibicategory (a bicategory without identity 1-cells).

We can do better:

- we can discover structures hidden by these particular specifics;
- we can put more formal category theory in the picture (à la Goguen, Guitart, van den Bril, Betti/Kasangian,. . .).

If you stare at the definition long enough, you'll notice that

$$\begin{array}{ccc} \mathit{Mly}(A,B) & \longrightarrow & A\otimes -/B \\ \downarrow & \lrcorner & \downarrow \\ \mathit{Alg}(A\otimes -) & \longrightarrow & K \end{array} \qquad \begin{array}{ccc} \mathit{Mre}(A,B) & \longrightarrow & K/B \\ \downarrow & \lrcorner & \downarrow \\ \mathit{Alg}(A\otimes -) & \longrightarrow & K \end{array}$$

(where $\mathit{Alg}(A\otimes -)$ is the category of endofunctor algebras

and up right there are comma categories)

If you stare even longer, you'll see $A \otimes -$ can be replaced with a left adjoint $F : K \to K$

$$\begin{array}{ccc} Mly(F, B) & \longrightarrow & F/B \\ \downarrow & \lrcorner & \downarrow \\ Alg(F) & \longrightarrow & K \end{array} \qquad \begin{array}{ccc} Mre(F, B) & \longrightarrow & K/B \\ \downarrow & \lrcorner & \downarrow \\ Alg(F) & \longrightarrow & K \end{array}$$

(with similar conventions for $Alg(F)$ and F/B)

Let **K** be a strict 2-category with all finite weighted limits.

Fix a 0-cell C, an endo-1-cell $f : C \to C$ and consider as building blocks of our theory

- the inserter $u : I(f, 1_C) \to C$ or 'object of algebras' for f;
- for every $b : B \to C$ the comma object C/b (equipped with its canonical projection $C/b \to C$);
- the comma object $(f/b) \to C$.

$$\begin{array}{ccc} \mathbf{Mly}(f,b) & \longrightarrow & (f/b) \\ \downarrow & \lrcorner & \downarrow \\ I(f,1_C) & \longrightarrow & C \end{array} \qquad \begin{array}{ccc} \mathbf{Mre}(f,b) & \longrightarrow & C/b \\ \downarrow & \lrcorner & \downarrow \\ I(f,1_C) & \longrightarrow & C \end{array}$$

Let **K** be a strict 2-category with all finite weighted limits.

Consider objects $X, B \in \mathbf{K}$ in a diagram of the following form:

$$X \xrightarrow[1]{} X \xleftarrow[f]{} X \xrightarrow[f]{} X \xleftarrow[b]{} B$$

this is nothing but a certain (Cat-enriched) sketch of which Mealy/Moore automata are the models in **K**.

(link w/ Petrișan 'sketch of automata')

Advantages:

- it's tidy;
- clarifies that (in a sense) 'computational machines' are models for a limit sketch;

$\rightsquigarrow$ One has analogues for $\mathbf{Mly}(A, B)$, $\mathbf{Mre}(A, B)$ enriched over a quantale like $[0, \infty]^{op}$: it makes sense to consider a *metric space* $\mathbf{Mly}_{(X,d)}(f, b)$ associated to every nonexpansive map $f : X \to X$ and point $b \in X$.

monoidal automata $\to$ bicategorical automata

Automata in bicategories

A monoidal category *is just*™ a bicategory with a single object.

But then, do the definition given above make sense when instead of K we consider a bicategory $\mathbb{B}$ with more than one object?

This idea is not *entirely* new; it resembles old (and obscure) work of Bainbridge, modeling the state space of abstract machines as a functor, of which one can take the left/right Kan extension along an 'input scheme'. See work of Petrișan et al.

Definition

Let $\mathbb{B}$ be a bicategory; a bicategorical Moore (biMoore) machine in $\mathbb{B}$ is a diagram of 2-cells

$$e \overset{\sigma}{\Longleftarrow} e \circ i, e \overset{\delta}{\Longrightarrow} o$$

between 1-cells e, i, o.[1]

The fact that this span exists, *coherces the types* of i, o, e in such a way that i must be an endomorphism of an object A.

$$A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A \xrightarrow{i} A, \dots$$

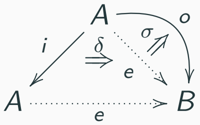
all make sense.

In the monoidal case, the fact that an input 1-cell stands on a different level from an output was completely obscured by the fact that every 1-cell is an endomorphism.

[1] A 1-cell of states (états), of inputs, and of outputs.

The terminal objects of $Mly(A, B), Mre(A, B)$ are respectively $[A^+, B], [A^*, B]$.

Analogously, given that a biMoore of fixed input and output i, o consists of a way of filling the dotted arrows in



with 1- and 2-cells, we have

The terminal object of the category of biMoore machines[2] is the right extension of $o : A \to B$ along the free monad $i^\sharp : A \to A$.

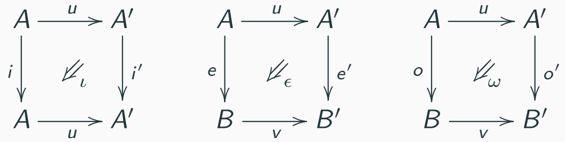
[2]With the obvious choice of morphisms, *mutatis mutandis*.

Intertwiners

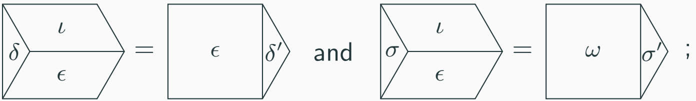
Definition (Intertwiner between bicategorical machines)

Consider two bicategorical Mealy machines $(e, \delta, \sigma)_{A,B}, (e', \delta', \sigma')_{A',B'}$ on different bases.

An *intertwiner* $(u, v) : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ consists of a pair of 1-cells $u : A \to A', v : B \to B'$ and a triple of 2-cells ι, ϵ, ω disposed as



such that



Intertwiners

Back to the monoidal (=one object) case, we obtain the following:

An intertwiner between (monoidal) machines $(E, d, s)_{I,O}$ and $(E', d', s')_{I',O'}$ consists of a pair of objects $U, V \in \mathcal{K}$, such that

1. there exist morphisms $\iota : I' \otimes U \to V \otimes I, \epsilon : E' \otimes U \to V \otimes E, \omega : O' \otimes U \to V \otimes O$;
2. the following two identities hold:

$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$
$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

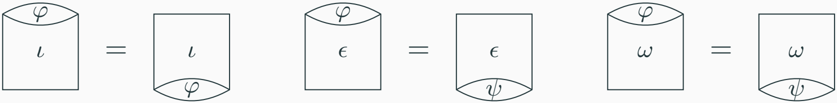
This notion is *not* trivial in the monoidal case!

Intertwiner 2-cells

Intertwiners between machines support a notion of higher morphisms:

Definition (2-cell between machines)

Let $(u, v), (u', v') : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ be two parallel intertwiners; a 2-cell $(\varphi, \psi) : (u, v) \Rightarrow (u', v')$ consists of a pair of 2-cells $\varphi : u \Rightarrow u'$, $\psi : v \Rightarrow v'$ such that



This notion is *not* trivial in the monoidal case!

Conclusions

Monoidal topology and automata

Let $T : \mathrm{Set} \to \mathrm{Set}$ be a monad, and $\mathcal{V}$ a quantale.

Clementino, Hofmann, Seal, Tholen... build locally thin bicategories of $(T, \mathcal{V})$-matrices and $(T, \mathcal{V})$-categories providing a unified description of the categories of topological spaces, approach spaces, metric and ultrametric, probabilistic-metric closure spaces...

BiMoore and biMealy machines, when instantiated in $(T, \mathcal{V})$-Prof, a 2-categorical way to look at topological, (ultra)metric ways to study behaviour of a state machine.

The reachability relation becomes topological, (ultra)metric, probabilistic, sequential... according to suitable choices of $T, \mathcal{V}$.