Category Theory (ITI9200) – Exercise Sheet 2

Outline. Complete as many of the exercises below as you are able. Each exercise has a number of tasks. Each task has an assigned number of points in square brackets, e.g. **[1]**. Points may be awarded for answers that demonstrate effort, even if the answer is not entirely correct. There are **25** total points (3 are points for overachievers, the exercise is marked as **[3]***). The exercise sheet is expected to take around 2 – 4 hours.

Submission. Email your work to Fosco Loregian at

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or hand in your work to one of the lecturers or teaching assistants at the start of end of a lecture. Deadline:

23:59 on April 18, 2024.

Exercise 1

Let **Set** denote the category of sets and total functions. Define the morphism part, and verify that the following correspondences given at the level of objects define functors **Set** $\to$ **Set** (show that the functor laws are satisfied):

- fix a set A; send a set X to $A + (X \times A)$ **[1]**
- send a set X to $PX \times \{0,1\}$ where PX is the set of subsets of X, and $\{0,1\}$ is a set of Boolean values. **[1]**

Define the morphism part, and verify that the following correspondences define functors (show that the functor laws are satisfied):

- Let $\mathcal{C}$ be the category where objects are pairs (X, R): X is a set, $R \subseteq X \times X$ a relation on X. Morphisms in $\mathcal{C}((X,R),(Y,S))$ are the functions $f : X \to Y$ such that if $(x, x') \in R$ then $(fx, fx') \in S$. Define a functor $\Gamma : \mathcal{C} \to$ **Graph** sending an object (X, R) of $\mathcal{C}$ to the graph having X as set of vertices, and an edge $x \rightsquigarrow x'$ if and only if $(x, x') \in R$. **[2]**
- Let **Graph** be the category of graphs and graph homomorphisms. If $\mathcal{G}$ is a graph, define the set $Sub(\mathcal{G})$ as the set of *subgraphs* of $\mathcal{G} = (\mathcal{G}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathcal{G}_0)$, i.e. the graphs $\mathcal{H}$ such that
  - $\mathcal{H}_0$ (the vertices of $\mathcal{H}$) is a subset of $\mathcal{G}_0$, and $\mathcal{H}_1$ (the edges of $\mathcal{H}$) is a subset of $\mathcal{G}_1$;
  - for every $e \in \mathcal{H}_1$, $s(e), t(e) \in \mathcal{H}_0$.

  Sending $\mathcal{G}$ to the set of all its subgraphs $\mathcal{H} \subseteq \mathcal{G}$ is a functor $Sub : \mathbf{Graph}^{\mathrm{op}} \to \mathbf{Set}$. **[2]**

(Trying to engage with this last question is totally optional, but if you try and get it wrong, there will be no negative repercussion. Do your best!)

Find a graph $\mathcal{W}$ such that $Sub(\mathcal{G})$ is parametrically isomorphic to the set $\mathbf{Graph}(\mathcal{G}, \mathcal{W})$. **[3]***

Exercise 2

Let A, B, C be sets.

Define functions in opposite directions:

$$A \times (B + C) \xrightarrow{p} A \times B + A \times C \qquad A \times B + A \times C \xrightarrow{q} A \times (B + C)$$

that are inverse to each other: $q \circ p = 1_{A\times(B+C)}$ and $p \circ q = 1_{A\times B+A\times C}$ (verify that these identities hold after defining p, q). **[2]**

In any category $\mathcal{C}$ with products and sums, it is possible to define a morphism

$$A \times B + A \times C \to A \times (B + C)$$

for each triple of objects $A, B, C \in \mathcal{C}_0$. $\mathcal{C}$ is called **distributive** if this morphism is an isomorphism for all $A, B, C \in \mathcal{C}_0$. Verify that in a distributive category $\mathcal{C}$ that admits an initial object $\varnothing$, the product $A \times \varnothing$ is an initial object (verify that $A \times \varnothing$ has the universal property of an initial object). **[2]**

Exercise 3

Let **Set** be the category of sets and total functions, and E a fixed set (a set of 'errors').

- Recalling that $\mathbf{Maybe}(A) = 1 + A$ denotes the sum of the sets A and $1 = \{\star\}$, use distributivity to expand the definition of the correspondence $\epsilon : \mathbf{Set} \to \mathbf{Set}$ sending a set X of 'states' to the set $\mathbf{Maybe}(X \times \mathbf{Maybe}(E))$. **[1]**
- Verify in detail that ϵ is a functor. **[1]**
- An 'ϵ-algebra' is a function

  $$\epsilon(X) \xrightarrow{\quad a \quad} X.$$

  Give an intuition for what a ϵ-algebra might represent, i.e. "It's a function sending an input in ϵX to..." **[2]**

  Prove that the set $(1 + E)^* = \mathbf{List}(\mathbf{Maybe}(E))$ has a structure of an ϵ-algebra $\xi : \epsilon(1 + E)^* \to (1 + E)^*$, and that, for every ϵ-algebra (X, a), there exists a unique function $h : (1 + E)^* \to X$ such that

  $$\begin{array}{ccc} \epsilon(1+E)^* & \xrightarrow{\quad \epsilon h \quad} & \epsilon(X) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle a} \\ (1+E)^* & \xrightarrow[\quad h \quad]{} & X \end{array}$$

  is a commutative square of functions. **[2]**

Exercise 4

Let **Dyn** be the category having

- objects the triples (X, x_0, f) where X is a set, $x_0 \in X$ is an element, and $f : X \to X$ is a function;
- a morphism $(X, x_0, f) \to (Y, y_0, g)$ is a function $h : X \to Y$ such that
  - $h(x_0) = y_0$;
  - $h \circ f = g \circ h$ (which means, the two composed functions coincide input-wise);
- composition and identities are defined as composition of functions in **Set**, as expected.

The category **Dyn** has an initial object $(\mathbb{N}, z, s)$ where N is the set of natural numbers $\{0, 1, 2, \dots\}$. What are z and s? Show that this choice of z and s does define an initial object. **[2]**

Verify that the category **Dyn** is the category of algebras for the endofunctor $\mathbf{Maybe}(-) : \mathbf{Set} \to \mathbf{Set}$ (where $\mathbf{Maybe}(X) = 1 + X$). **[1]**

Show that we can use the fact that $(\mathbb{N}, z, s)$ is initial to define functions, for each fixed $p \in \mathbb{N}$,

$$\text{plus}_p : N \to N$$
$$\text{times}_p : N \to N$$

such that, for all natural numbers $x \in \mathbb{N}$, the following equations hold:

$$\text{plus}_p(x) = p + x$$
$$\text{times}_p(x) = p \times x$$

[3]