The Rosen Fibration

ACT26 — Tallinn

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How it started

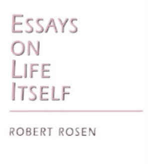
At the end of October 2025, Amahury (L) and Sean (R, with doggo) contacted me.

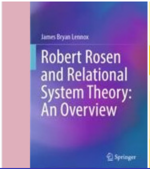
Intro

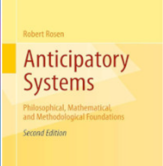
- Applying category theory to biology: ‘vindicating’ Robert Rosen, who believed category theory was a foundation for life sciences.
- His work foreshadows ideas in endofunctor algebra theory, monads, systems and processes—concepts we understand only now.
- This project (with Amahury and Sean) is a work in progress; comments welcome.
- This is a talk about mathematics.

- Attended lectures of Eilenberg, then Mac Lane
- Wrote extensively on category theory in biology
- not always correct...

Wrote before Yoneda lemma, adjunctions, monads . . . and stayed outside the category theory community.







MR (or '(M,R)') systems

A 'living system' is capable of

M) etabolism (f)

R) eparation (Φ)

$$A \xrightarrow{f} B \xrightarrow{\Phi} [A, B]$$

- f converts reagents A into products B;
- Φ is a higher-order function generating 'function types' $[A, B]$, counteracting the withering of f.

A candle has f but not Φ; Ni-Ti alloy has Φ (seals microcracks) but no metabolism.

MR systems

Rosen works in a subcategory of sets;

core idea: there is an intrinsic self-referentiality property for alive systems ("closure under efficient causation": $\exists \bar{a}.\, \lambda x.\Phi(f\bar{a})x = \lambda x.fx$) but lacks monoidal closed categories (1958).

Note: foreshadows Lawvere fixpoint. Note: Cartesianity is unduly strong in biology! Life is a linear phenomenon.

Our goal: study the diagram $A \to B \to [A, B]$ in a suitable sense. Can we build a category

$$\sum_{A,B:\mathcal{C}} \left\{ A \xrightarrow{f} B \xrightarrow{\Phi} [A, B] \right\}?$$

Yes — **MRS** is the Rosen fibration of the title.

Incoherent systems

Working Assumption

$\mathcal{C}$ is symmetric monoidal closed, with all needed (co)limits.

Definition

An incoherent MR system is $A \xrightarrow{f} B \xrightarrow{\Phi} [A, B]$ in $\mathcal{C}$. A morphism $(\ell, r) : (f, \Phi) \rightharpoonup (g, \Psi)$ makes

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "l"'] & B \arrow[r, "\Phi"] \arrow[d, "r"'] & {[A,B]} \arrow[dr, "{[A,r]}"] & \\
 & & & {[A,B']} \\
A' \arrow[r, "g"'] & B' \arrow[r, "\Psi"'] & {[A',B']} \arrow[ur, "{[\ell,B']}"'] &
\end{tikzcd}$$

commute. The total category **iMRS** has triples $(A, B; f, \Phi)$ as objects and morphisms as above.

I use agda, btw

```
record iMR2 (A B : Obj) : Set (o ⊔ ℓ) where
  constructor ⟪_,_⟫
  field
    f : A ⇒ B
    ϕ : B ⇒ [ A , B ]₀

record iMR2₀ : Set (o ⊔ ℓ) where
  field
    A B : Obj
    ξ : iMR2 A B

record iMR2⇒ (X Y : iMR2₀) : Set (o ⊔ ℓ ⊔ e) where
  module X = iMR2₀ X
  module Y = iMR2₀ Y
  module ξX = iMR2 X.ξ
  module ξY = iMR2 Y.ξ
  field
    l : X.A ⇒ Y.A
    r : X.B ⇒ Y.B
    eqf : r ∘ ξX.f ≈ ξY.f ∘ l
    eqϕ : [ l , id ]₁ ∘ ξY.ϕ ∘ r ≈ [ id , r ]₁ ∘ ξX.ϕ
```

Reasoning fiberwise

(✋) Fix A: category $\mathbf{iMRS}^A$ ($\ell = \mathrm{id}$)

- same A, varying B,
- morphisms $v : B \to B'$ with

$$\begin{tikzcd}
 & B \arrow[r, "\Phi"] \arrow[d, "v"'] & {[A,B]} \arrow[d, "{[A,v]}"] \\
A \arrow[ur, "f"] \arrow[r, "g"'] & B' \arrow[r, "\Psi"] & {[A,B']}
\end{tikzcd}$$

```
record iMR2ᴿ₀ (A : Obj) : Set (o ⊔ ℓ ⊔ e) where
  field
    B : Obj
    ξ : iMR2 A B

record iMR2ᴿ⇒ {A : Obj} (X Y : iMR2ᴿ₀ A) : Set (o ⊔ ℓ ⊔ e) where
  module X = iMR2ᴿ₀ X
  module Y = iMR2ᴿ₀ Y
  module ξX = iMR2 X.ξ
  module ξY = iMR2 Y.ξ
  field
    v : X.B ⇒ Y.B
    eqf : v ∘ ξX.f ≈ ξY.f
    eqϕ : [ id , v ]₁ ∘ ξX.ϕ ≈ ξY.ϕ ∘ v
```

(✋) Fix B: category $\mathbf{iMRS}_B$ ($r = \mathrm{id}$)

- same B, varying A,
- morphisms $u : A \to A'$ with

$$\begin{tikzcd}
A \arrow[dr, "f"] \arrow[dd, "u"'] & & {[A,B]} \\
 & B \arrow[ur, "\Phi"] \arrow[dr, "\Psi"'] & \\
A' \arrow[ur, "g"'] & & {[A',B]} \arrow[uu, "{[u,B]}"']
\end{tikzcd}$$

```
record iMR2ᴸ₀ (B : Obj) : Set (o ⊔ ℓ ⊔ e) where
  field
    A : Obj
    ξ : iMR2 A B

record iMR2ᴸ⇒ {B : Obj} (X Y : iMR2ᴸ₀ B) : Set (o ⊔ ℓ ⊔ e) where
  module X = iMR2ᴸ₀ X
  module Y = iMR2ᴸ₀ Y
  module ξX = iMR2 X.ξ
  module ξY = iMR2 Y.ξ
  field
    u : X.A ⇒ Y.A
    eqf : ξX.f ≈ ξY.f ∘ u
    eqϕ : ξX.ϕ ≈ [ u , id ]₁ ∘ ξY.ϕ
```

Incoherent systems

Equivalent characterization of an incoherent MR system:

- Fix A; one can think of the category of iMRS 'based at A' as the category of algebras for the endofunctor of $\mathcal{C}$
$$X \mapsto A \otimes (I + X)$$
- Fix B; the category of iMRS 'based at B' is the slice $\mathcal{C}/[I + B, B]$.

A slightly more refined characterization, that works only in a distributive Cartesian base, is the following:

- An iMR based at A is a parametric, pointed dynamical system of type $\xi \approx \left(1 \overset{f}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} B; B \overset{\Phi}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} B \right)$ with certain coKleisli maps;
- More simply, an iMR seems to be a lens $\binom{A}{B} \to \binom{B}{B}$. (There is more to this story I haven't looked into...)

There is a problem!

$A \mapsto \mathbf{iMRS}^A$ is functorial (reindexing $v^* : \mathbf{iMRS}^{A'} \to \mathbf{iMRS}^A$).

$B \mapsto \mathbf{iMRS}_B$ is not functorial; the categories $\mathbf{iMRS}_B$ are displayed, not fibered, over $\mathcal{C}$.

```
left : (u : A ⇒ A') → Functor (iMR2ᴿ A') (iMR2ᴿ A)
left {A} {A'} u = record
  { F₀ = λ { x →
    let module x = iMR2ᴿ₀ x
    in record
    { B = x.B
    ; ξ = ⟪ iMR2.f x.ξ ∘ u , [ u , id ]₁ ∘ iMR2.ϕ x.ξ ⟫
    }}
  ; F₁ = λ { {x} {y} f →
      let module x   = iMR2ᴿ₀ x
          module ξx  = iMR2 x.ξ
          module y   = iMR2ᴿ₀ y
          module ξy  = iMR2 y.ξ
          module f = iMR2ᴿ⇒ f
      in record
    { v = f.v
    ; eqf = begin f.v ∘ f.ξX.f ∘ u   ≈⟨ sym-assoc ⟩
                  (f.v ∘ f.ξX.f) ∘ u ≈⟨ f.eqf ⟩∘⟨refl ⟩
                  f.ξY.f ∘ u         ∎
    ; eqϕ = begin [ id , f.v ]₁ ∘ [ u , id ]₁ ∘ f.ξX.ϕ ≈⟨ sym-assoc • ([ [-,-] ]-commute ⟩∘⟨refl) • assoc ⟩
                  [ u , id ]₁ ∘ [ id , f.v ]₁ ∘ f.ξX.ϕ ≈⟨ (refl⟩∘⟨ f.eqϕ) • sym-assoc ⟩
                  ([ u , id ]₁ ∘ f.ξY.ϕ) ∘ f.v ∎
    }}
  ; identity = λ {A} → refl
  ; homomorphism = λ {X} {Y} {Z} {f} {g} → refl
  ; F-resp-≈ = λ x → x
  }
```

Definition

A coherent MR system based at $A, B \in \mathcal{C}_0$ consists of

- $f : A \to B$ in $\mathcal{C}$;
- a natural transformation [cf. [a] and [b]] $\Phi : cod \Rightarrow [A, cod]$, where $cod : \mathcal{C}^{[0\to 1]} \to \mathcal{C}$ is the codomain functor.

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[a] Ion C. Baianu, Natural transformations of organismic structures, Bulletin of Mathematical Biology, Volume 42, Issue 3, 1980

[b] Baianu, I. and Marinescu, M. 1974. A Functorial Construction of (M,R)-Systems. Rev. Roum. Math. Pures et Appl., 19

any mapping $t : X \to Y$ into the same mapping t of Set. The replication maps Φ_f are now defined by the *natural transformations* $\Phi : I \to h^X$, with X varying in **M**. Let us denote by (h^X, I)-the sets of all natural

Coherent MR systems

Φ is pointed by f; naturality means

$$\begin{tikzcd} Y \arrow[r, "\Phi_h"] \arrow[d, "r"'] & {[A, Y]} \arrow[d, "{[A,r]}"] \\ Y' \arrow[r, "\Phi_k"] & {[A, Y']} \end{tikzcd}$$

commutes whenever $\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\ell"'] & Y \arrow[d, "r"] \\ X' \arrow[r, "k"'] & Y' \end{tikzcd}$ is an arrow $h \to k$ in the category $\mathcal{C}^{[0\to 1]}$.

Towards the Rosen fibration

The profunctor of MR systems |

MRS(A, B) is a profunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$; its action on morphisms $u : A' \to A$, $v : B \to B'$ is

$$\begin{array}{ccc} \mathbf{MRS}(A, B) & \longrightarrow & \mathbf{MRS}(A', B') \\ (A \to B, \Phi) & \longmapsto & (A' \to A \to B \to B', [u, cod] \circ \Phi) \end{array}$$

```
MRS-Profunctor : Bifunctor (Category.op C) C (Setoids (o ⊔ ℓ ⊔ e) (o ⊔ ℓ ⊔ e))
MRS-Profunctor = record
  { F₀ = λ { (A , B) → MR2-Setoid A B }
  ; F₁ = λ { {(A , B)} {(A' , B')} (u , v) → record
    { _⟨$⟩_ = λ {⟪ f , ϕ ⟫ → ⟪ v ∘ f ∘ u , (nHom u ∘ʳ Cod) ∘ᵥ ϕ ⟫ }
    ; cong = λ { {⟪ f , ϕ ⟫} {⟪ g , ϕ' ⟫} (f≈g , ϕ≈ϕ') →
        (∘-resp-≈ Equiv.refl (∘-resp-≈ f≈g Equiv.refl))
      , (λ {x} → ∘-resp-≈ʳ (ϕ≈ϕ' {x}))
      }
    }}
```

To the profunctor **MRS** one can associate several categories:

- The tabulator and cotabulator: the universal categories with cells

$$
\begin{tikzcd}
 & \oslash\mathbf{MRS} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\overset{\tau}{\Rightarrow}"] & \\
\mathcal{C} \arrow[rr, "\mathbf{MRS}"'] & {} & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathbf{MRS}"] \arrow[dr] & {} \arrow[d, phantom, "\overset{\kappa}{\Rightarrow}"] & \mathcal{C} \arrow[dl] \\
 & \oslash\mathbf{MRS} &
\end{tikzcd}
$$

- The category of elements $\mathcal{E}(\mathbf{MRS})$ via pullback;

The two-sided discrete fibration $\mathcal{C} \leftarrow \oslash\mathbf{MRS} \rightarrow \mathcal{C}$ is the Rosen fibration of the title.

Arbib[1] showed each MR system determines a Mealy automaton (input A, output B, state $[A, B]$), functorially:

There is a total category of Mealy automata and a fibred functor

$$
\begin{tikzcd}
 & \mathcal{C} & \\
\owedge\mathbf{MRS} \arrow[ur, "p_1"] \arrow[dr, "p_2"'] \arrow[rr, "\Gamma"] & & \mathbf{Mly} \arrow[ul, "q_1"'] \arrow[dl, "q_2"] \\
 & \mathcal{C} &
\end{tikzcd}
$$

[1] M. A. Arbib, "Categories of (M-R) systems". Bull. Math. Biophysics 28, 511-517 (1966).

Fibration of repair maps

The presheaf $Rep : \mathcal{C}^{op} \to \mathbf{Set}$, $A \mapsto \mathbf{Cat}(\mathcal{C}^{[0\to 1]}, \mathcal{C})(cod, [A, cod])$ has the fibration of repairs $\mathcal{E}(Rep)$ as its category of elements.

Objects: $(A, \Phi : cod \Rightarrow [A, cod])$; **morphisms** $u : (A, \Phi) \to (A', \Upsilon)$ making $\Phi = \Upsilon * [u, cod]$.

$$\begin{tikzcd} & cod \arrow[dl, "\Upsilon"'] \arrow[dr, "\Phi"] & \\ {[A', cod]} \arrow[rr, "{[u,cod]}"'] & & {[A, cod]} \end{tikzcd}$$

The fibration of repairs is a discrete, coreflective subfibration of the tabulator:

$$\begin{tikzcd} \mathcal{E}(Rep) \arrow[r, hook, shift right, "\mathbb{J}"'] \arrow[r, phantom, "\top"] & \mathcal{T}(\mathbf{MRS}) \arrow[l, shift right, "k"'] \end{tikzcd} \qquad k\big(A, B; \xi @ (f, \Phi)\big) = (A, \Phi)$$

Longer MRs

Rosen and others [2] [3] [4] considered longer MR systems

$$A \longrightarrow B \longrightarrow [A, B] \longrightarrow [B, [A, B]] \longrightarrow \dots$$

defined by a Fibonacci rule $M_0 = A,\ M_1 = B,\ M_n = [M_{n-2}, M_{n-1}]$.

[2] M. W. Warner, "Representations of (M,R)-systems by categories of automata". Bull. Math. Biology 44, 661–668 (1982).

[3] R. Rosen, "Anticipatory Systems..." Pergamon Press (1985).

[4] A. H. Louie, "More than life itself..." Ontos Verlag (2009).

What is a 3-step MR system? Two MR systems in a trenchcoat (sewn along f and Φ):

$$
\begin{tikzcd}
A \arrow[r] & B \arrow[r, "R(f,\Phi)"] \arrow[d, equal] & {[A,B]} \arrow[d, equal] & \\
 & X \arrow[r, "M(g,\Psi)"'] & Y \arrow[r] & {[X,Y]}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{MRS}_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}(\mathbf{MRS}) \arrow[d, "M"] \\
\mathcal{E}(\mathbf{MRS}) \arrow[r, "R"'] & \mathcal{C}^{[0\to 1]}
\end{tikzcd}
$$

R ('choose Φ') exists only with domain the category of elements!

$$\mathbf{MRS}_\infty := \lim\left(\overset{:=\owedge\mathbf{MRS}}{\mathbf{MRS}_2} \longleftarrow \mathbf{MRS}_3 \longleftarrow \dots \longleftarrow \mathbf{MRS}_n \longleftarrow \dots \right)$$

This 'vindicates Rosen': it makes *precise*, in *modern terms*, an idea he had decades ago.

Open questions

Let me conclude.

- An MR system is a datatype; study it.
- Find examples: what is an MR system of vector spaces? is $[n] \mapsto \mathbf{MRS}_n$ a 2-Segal space? what if $A \to B \to [A, B]$ is a chain complex?
- If $\mathcal{C}$ is Cartesian, **MRS** is a pro-comonad; what is a Rosen pro-coalgebra?
- Formalise everything in Agda:

agda-categories/tree/rosen/src/Categories/Rosen

Thanks!

Speculations on the big picture

- Using structural mathematics to study “living” systems is an old idea;
- It remains unclear whether biology even admits a foundational treatment in terms of axioms and theorems (Schiaparelli, Freguglia. . .);
- “Living” systems interact with their environment through dual feedback loops, and exhibit self-repair, energy storage, resilience, and many other functions;
- *While category-theoretic frameworks exist for modeling feedback, resource theories, and other aspects individually, there is currently no framework capable of integrating all these dimensions into a single mathematical theory.*
- Any non-trivial contribution to this problem should be regarded as a valuable advance for both mathematics and the life sciences;
- To me, as a mathematician, one of Nature’s purposes is to inspire profound mathematical questions —a perspective that is often overlooked.