Kan extensions

Topics in category theory – Brno 2018

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Motivations

Many problems in Algebra and Topology can be subsumed by the following general scheme:

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

a “calculus of lifting problems and their solution”.

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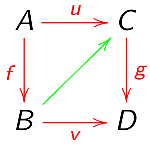


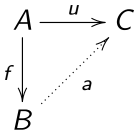
a "calculus of lifting problems and their solution".

Motivations

We say that f is left orthogonal to g, or g is right orthogonal to f, if for every such commutative square there is a unique $a : B \to C$ such that the two triangles commute. This calculus arises, like, everywhere.

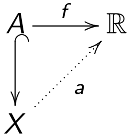
- factorization systems (which you will see) are based on this definition; and
- model categories (if you like algebraic topology look into the definition)

Particularly ubiquitous are problems where $D = *$ is a terminal object, e.g. where have to find a solution to an extension problem:

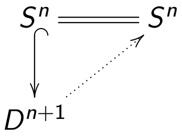


Motivations

- Tietze extension theorem: let $f : A \to \mathbb{R}$ be Lipschitz continuous and defined on a subspace of a metric space $A \subseteq X$; then there exists an extension (non unique) $a : X \to \mathbb{R}$,



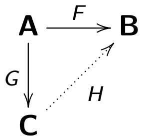
  with the same Lipschitz constant of f.
- Brouwer theorem, categorically: given a sphere $S^n \subseteq D^{n+1}$



  there is no continuous retraction $r : D^{n+1} \to S^n$.

Motivations

To solve a similar problem in the category of categories, i.e. given a diagram

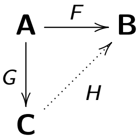


of categories and functors, we face an intrinsic difficulty.

There are many more degrees of freedom given by the fact that in **Cat** there are also natural transformations that can fill the diagram strictly, strongly or laxly.

Kan extensions

A reasonable request would be: is there a way to fill



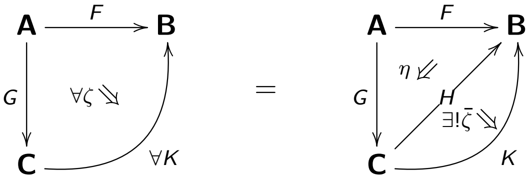
with a natural transformation $\eta : F \Rightarrow HG$? Of course, we do not want to loosen the problem too much, so we might as well ask for some sort of uniqueness to η.

Definition

We can "solve the extension problem at the pair (G, F)" if there exists a third functor $H : \mathbf{C} \to \mathbf{B}$ with a natural transformation $\eta : F \to HG$ which is initial among all such pairs $\langle K, \xi \rangle$.

Kan extensions

This means that



Definition

We say that the pair $\langle H, \eta \rangle$ "exhibits $\mathrm{Lan}_G F$" when it has this property; we often say simply that "H is $\mathrm{Lan}_G F$".

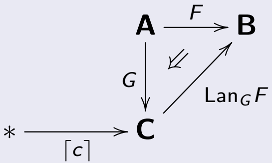
pointwise Kan extensions

It turns out that this definition, albeit correct, it too general.

To save the situation we can give a less general one encompassing many (actually, all) examples of interest. This leads to the definition of pointwise Kan extension

Idea

It would be nice to compute $\mathrm{Lan}_G F$ "locally", in the following way: given $c \in \mathbf{C}$



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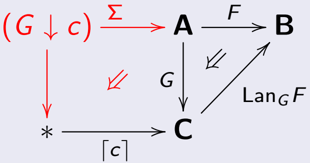

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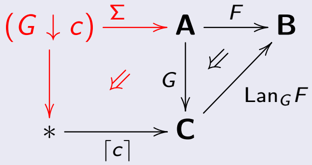

and $\mathrm{Lan}_G F(c) = \mathrm{Lan}_G F \circ \lceil c \rceil \cong \mathrm{colim}_{(G \downarrow c)} F \circ \Sigma$.

It turns out that this simple idea actually works.

Proposition

When the colimit

$$\operatorname{colim}_{(G\downarrow c)} F \circ \Sigma$$

exists for every $c \in \mathbf{C}$, the assignment $c \mapsto \operatorname{colim}_{(G\downarrow c)} F \circ \Sigma$ can be promoted to a functor $\mathbf{C} \to \mathbf{B}$ that has the same universal property of $\mathrm{Lan}_G F$ in the category $[\mathbf{C}, \mathbf{B}]$.

This special instance of $\mathrm{Lan}_G F$ is called the "pointwise" left Kan extension of F along G, to reinforce the intuition that the value of $\mathrm{Lan}_G F$ can be computed "on points" of its domain.

pointwise Kan extensions

Definition

A functor $P : \mathbf{B} \to \mathbf{X}$ *preserves* $\mathrm{Lan}_G F$ if $P \circ \mathrm{Lan}_G F \cong \mathrm{Lan}_G PF$.

Proposition

Let $\mathbf{X} = \mathbf{Set}$ and $P = \mathbf{B}(-, X) : \mathbf{B} \to \mathbf{Set}$ a representable presheaf. Then P preserves every pointwise Kan extension.

$$\begin{aligned} P \circ \mathrm{Lan}_G F &\cong \hom(\mathrm{Lan}_G F(C), X) \cong \hom(\operatorname*{colim}_{A:G\downarrow c} FA, X) \\ &\cong \lim_{A:G\downarrow c} \hom(FA, X) \cong \lim_{A:G\downarrow c} P(FA) \cong \mathrm{Lan}_G PFC \end{aligned}$$

Is the converse true?

A fundamental theorem

Let $\mathrm{Lan}_G F$ be pointwise. The the correspondence $F \mapsto \mathrm{Lan}_G F$ can be promoted to a functor

$$\mathrm{Lan}_G : [\mathbf{A}, \mathbf{B}] \to [\mathbf{C}, \mathbf{B}]$$

and there is an adjunction $\mathrm{Lan}_G \dashv G^*$ (where $G^*(K) = K \circ G$).

There are many ways to prove this statement; we opt for an hands-on proof, defining a natural bijection

$$[\mathbf{C}, \mathbf{B}](\mathrm{Lan}_G F, H) \leftrightarrows [\mathbf{A}, \mathbf{B}](F, HG)$$

proving also that η is the unit of the adjunction.

Haskell-y remark

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Lan_G   :: (A -> B) -> (C -> B) --this is a left adjoint
Lan_G F :: C -> B --this is not (always)
```

A fundamental theorem

Exercise

Prove the above proposition.

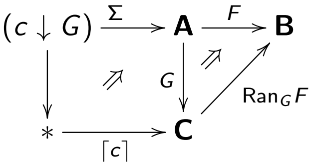
Exercise

Prove the dual of the above "fundamental theorem" for right Kan extensions, defined below.

Exercise

What is the colimit expressing the pointwise extension $\mathrm{Lan}_1 F$? Relate this result with the Yoneda lemma.

- A right Kan extension of F along G is a pair $\langle \mathrm{Ran}_G F : \mathbf{C} \to \mathbf{B}, \epsilon : \mathrm{Ran}_G F \circ G \to F \rangle$ terminal among all such pairs.
- A pointwise right Kan extension is obtained from the limit



- There is an adjunction $G^* \dashv \mathrm{Ran}_G$ with counit ϵ.

In nice situations (for example when $\mathbf{B}$ is complete and cocomplete) every functor $G : \mathbf{A} \to \mathbf{C}$ defines a triple of adjoints

$$\mathrm{Lan}_G \dashv G^* \dashv \mathrm{Ran}_G : \; [\mathbf{C}, \mathbf{B}] \begin{array}{c} \xleftarrow{\mathrm{Ran}_G} \\ \longrightarrow \\ \xleftarrow[\mathrm{Lan}_G]{} \end{array} [\mathbf{A}, \mathbf{B}]$$

Examples

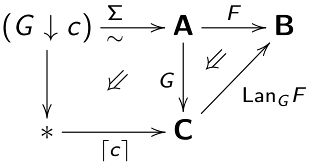
Limits and colimits

All colimits are Kan extensions: let $\mathbf{C} = *$ be the terminal category and $G = ! : \mathbf{A} \to *$ the terminal functor.

Then the functor $(G \downarrow c) \to \mathbf{A}$ is an isomorphism and

$$\operatorname*{colim}_{(!\downarrow c)} F \circ \Sigma \cong \operatorname{colim} F$$

(because Σ is an equivalence; exercise: prove that an equivalence is a *cofinal* functor).



Dually, $\lim_{(c\downarrow !)} F\Sigma \cong \lim F$.

Adjoints

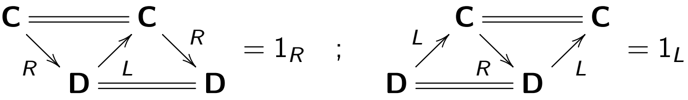
Under reasonable conditions, Kan extensions provide adjoints to functors.

Proposition

Let $R : \mathbf{C} \to \mathbf{D}$ be a functor. The following conditions are equivalent

- R has a left adjoint L with unit and counit $L \frac{\eta}{\epsilon}\!\!\mid R$.
- $\langle L, \epsilon \rangle$ exhibits $\mathrm{Ran}_R 1$ and R preserves this right Kan extension

The proof can be done elementarily (good luck) or with a moderate amount of 2-cell calculus, recalling the zigzag identities



for the unit and counit of an adjunction $L \frac{\eta}{\epsilon}\!\!\mid R$.

Induction and coinduction

Let $i : H \leq G$ be the inclusion of a subgroup regarded as a functor between one-objet categories; if k is a field, the category of k-linear representations of G can be identified with the functor category $[G, \mathbf{Vect}_k]$, and similarly for H.

The category of finite dimensional vector spaces has finite co/limits, so if G, H are finite the functor

$$[H, \mathbf{Vect}] \to [G, \mathbf{Vect}]$$

induced by i has both a left and a right adjoint by virtue of our fundamental theorem.

These adjoints, Lan_i and Ran_i, can be respectively computed as (V is a H-module)

$$V \mapsto k[G] \otimes_{k[H]} V \in G\text{-}\mathbf{Vect}\, V \mapsto \quad \hom_{k[H]}(k[G], V) \in G\text{-}\mathbf{Vect}$$

These, in turn, can be computed as the limit and colimit, resp.,

$$\hom_{k[H]}(k[G], V) \longrightarrow \hom_k(K[G], V) \rightrightarrows \prod_{h\in H} \hom(k[G], V)$$

$$\coprod_{h\in H} K[G] \otimes_k V \rightrightarrows k[G] \otimes_k V \longrightarrow k[G] \otimes_{k[H]} V$$

Where we take the equivariant linear maps of $k[H]$-modules and we quotient out for the relation prescribing $(h.\alpha) \otimes v - \alpha \otimes (h.v)$.

Sheaves

Let X be a topological space and $Op(X)$ the category of its open subsets. A presheaf on X is a functor $Op(X)^{\text{op}} \to \mathbf{Set}$. The category of sets is complete and cocomplete, so that every continuous function $f : Y \to X$ induces a functor

$$[Op(X)^{\text{op}}, \mathbf{Set}] \xrightarrow{f^*} [Op(Y)^{\text{op}}, \mathbf{Set}]$$

that has both a left and a right adjoint: $f_! \dashv f^* \dashv f_*$.

These adjoints are computed as follows:

$$\begin{cases} f_!(\mathcal{F})(V) = \text{colim}_{U \supseteq fV} \mathcal{F}(U) \\ f_*(\mathcal{F})(V) = \mathcal{F}(f^{\leftarrow} Vs) \end{cases}$$

Exercise: the first equation is the formula of our fundamental theorem: prove it.

Graphs

A directed graph is a covariant functor on the category $\Gamma = \{E \overset{s}{\underset{t}{\rightrightarrows}} V\}$ (a directed graph is a set of vertices, a set of edges, and two functions giving source and target). The "forgetful" functor $U : \mathbf{dGph} \to \mathbf{Set}$ is the functor

$$t_\Gamma^* : [\Gamma, \mathbf{Set}] \to \mathbf{Set} = [*, \mathbf{Set}]$$

where the functor $t : * \to \Gamma$ chooses the object V and its identity morphism. Since **Set** is co/complete, t_Γ^* has both a left and a right adjoint by virtue of our fundamental theorem.

Exercise

Show that Lan_{t_Γ} exists and it is given by the functor that sends a set X to the directed graph having X as set of vertices, and no edges.

What does the functor Ran_{t_Γ} do?

Nerves

Let $\mathbf{\Delta}$ be the category of nonempty finite totally ordered sets and monotone functions; denote $[n] = \{0 < 1 < \dots < n\}$ the generic object and $\Delta[n]$ the representable object on it. Let $J : \mathbf{\Delta} \to \mathbf{Cat}$ the tautological functors that regards a poset as a cateogry.

We can define a functor $\mathbf{Cat} \to [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$ (the category of simplicial sets) such that $[n]$ is sent to

$$N(\mathcal{C})_n = \mathbf{Cat}(J[n], \mathcal{C}) = \{C_0 \to C_1 \to \dots \to C_n \mid C_i \in \mathcal{C}\}$$

Proposition

The functor $\mathcal{C} \mapsto (\lambda[n].N(\mathcal{C})_n)$ has a left adjoint given by $\text{Lan}_y J : [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}] \to \mathbf{Cat}$, where $y : \mathbf{\Delta} \to [\mathbf{\Delta}^{\text{op}}, \mathbf{Set}]$ is the Yoneda embedding functor.

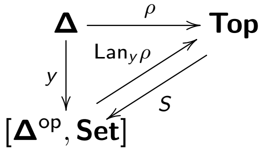
Nerves

We prove that $\mathrm{Lan}_y J \dashv N$ by providing a natural bijection of hom-sets: given a category $\mathcal{D}$ and a simplicial set X we have

$$\begin{aligned}
\mathbf{Cat}(\mathrm{Lan}_y J(X), \mathcal{D}) &\cong \mathbf{Cat}\left(\operatorname*{colim}_{([n],x)\in(y\downarrow X)} J\Sigma([n],x), \mathcal{D}\right) \\
&\cong \lim_{([n],x)\in(y\downarrow X)} \mathbf{Cat}\left(J\Sigma([n],x), \mathcal{D}\right) \\
&\cong \lim_{([n],x)\in(y\downarrow X)} \mathbf{Cat}\left(J[n], \mathcal{D}\right) \\
&\cong \lim_{([n],x)\in(y\downarrow X)} [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\Delta[n], N(\mathcal{D})) \\
&\cong [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](\operatorname*{colim}_{([n],x)\in(y\downarrow X)} \Delta[n], N(\mathcal{D})) \\
&\cong [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}](X, N(\mathcal{D}))
\end{aligned}$$

Nerves

By virtue of a similar argument it is possible to prove that there is an adjunction



where the functor $S(X)$ is the simplicial set $[n] \mapsto \mathbf{Top}(\rho[n], X)$ and $\rho[n]$ is the topological simplex

$$\rho[n] = \{\vec{X} \in \mathbb{R}^n \mid X_i \geq 0,\ \sum X_i \leq 1\}$$

Nerves

Of course there is a pattern here:

Theorem

Let

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle y}\downarrow & & \\ [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}] & & \end{array}$$

be an arrangement of functors where $\mathbf{C}$ is small and $\mathbf{D}$ is cocomplete.

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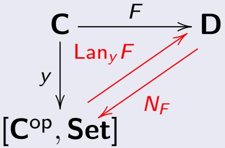

be an arrangement of functors where **C** is small and **D** is cocomplete. Then there is an adjunction $\mathrm{Lan}_y F \dashv N_F = \hom(F, 1)$ where F commutes with finite limits if and only if $\mathrm{Lan}_y F$ does.

Examples of nerves: Isbell duality

Isbell duality sets up an adjunction

$$[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \underset{Spec}{\overset{\mathcal{O}}{\rightleftarrows}} [\mathbf{A}, \mathbf{Set}]^{\mathrm{op}}$$

for every small category $\mathbf{A}$, defined as

$$\mathcal{O} : X \mapsto [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}](X, \hom(c, -))$$
$$Spec : Y \mapsto [\mathbf{A}, \mathbf{Set}](Y, \hom(-, c))$$

Riddle

Why are the functors called $\mathcal{O}$ ("ring of functions") and *Spec* ("spectrum")?

Examples of nerves: Isbell duality

Exercise

Show that

$$\begin{array}{ccc} & \mathbf{A} & \\ {}^{y}\swarrow & & \searrow^{y^o} \\ [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] & \underset{\mathrm{Lan}_{y^o}(y)}{\overset{\mathrm{Lan}_y(y^o)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & [\mathbf{A}, \mathbf{Set}]^{\mathrm{op}} \end{array}$$

is the Isbell adjunction.

Examples of nerves: space from sheaves

Let X be a topological space and $Op(X)$ the category of its open subsets. Define the tautological functor

$$\begin{aligned} Op(X) &\to \mathbf{Top}/X \\ U &\mapsto (U \subseteq X) \end{aligned}$$

Since $\mathbf{Top}/X$ is cocomplete, we can define an adjunction

$$[Op(X)^{\mathrm{op}}, \mathbf{Set}] \underset{sheafofsections}{\overset{etale}{\rightleftarrows}} \mathbf{Top}/X$$

$$\left(\begin{array}{ccc} U & \dashrightarrow & E \\ & {}_i\searrow \quad \swarrow_p & \\ & X & \end{array} \right) \hookleftarrow \begin{bmatrix} E \\ \downarrow \\ X \end{bmatrix}$$

that suitably restricted and corestricted determines the equivalence between *sheaves* and étale spaces over X.

A few references

- 3.7 of F. Borceux "Handbook of categorical algebra".
- Kashiwara and Schapira, section 2.3 in "Categories and Sheaves" (for an hands-on proof of the fundamental theorem).
- M. Vergura's notes (link) on Kan extensions.
- M. Lehner's B.Sc. thesis (link) "All concepts are Kan extensions"
- I.1–5 of Dubuc's "Kan extensions in enriched category theory".