Bicategories of automata, automata in bicategories

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Prolegomena

Fix a monoidal category $(\mathcal{K}, \otimes)$.

Definition

A Mealy machine (of input I and output O) in $\mathcal{K}$ is a span

$$E \xleftarrow{d} E \otimes I \xrightarrow{s} O$$

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A Moore machine (of input I and output O) in $\mathcal{K}$ is a span

$$E \xleftarrow{d} E \otimes I \,,\; E \xrightarrow{s} O$$

Prolegomena

```
record MealyObj I O : Set (o ⊔ l ⊔ e) where
  field
    E : Obj
    d : E ⊗ I ⇒ E
    s : E ⊗ I ⇒ O
```

Mealy.agda XMealy.agda Mealy/Bicategory.agda

Mealy and Moore

Definition

The category of Mealy machines[1] has objects the Mealy machines as above, (E, d, s), and morphisms $(E, d, s) \to (F, d', s')$ the $f : E \to F$ such that

$$\begin{array}{ccccc} E & \xleftarrow{d} & E \otimes I & \xrightarrow{s} & O \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes I} & & \| \\ F & \xleftarrow{d'} & F \otimes I & \xrightarrow[s']{} & O \end{array}$$

[1] All definitions from now on can be Moore-ified without effort.

Mealy and Moore

Let P be an monoidal monad on $\mathcal{K}$; the Kleisli category of P becomes monoidal; dually, if P is opmonoidal, Elienberg-Moore (not the same Moore!) becomes monoidal.

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Why care?

- nondeterminism (if $P =$powerset, $Kl(P) = \mathbf{Rel}$);
- additional structure on objects (they are P-algebras).

Mealy and Moore

Theorem

The category of Mealy machines fits into a (strict, 2-)pullback in **Cat**

$$\begin{array}{ccc} \mathbf{Mly}(I,O) & \longrightarrow & \mathbf{Alg}(-\otimes I) \\ \big\downarrow & \lrcorner & \big\downarrow \\ ((-\otimes I)/O) & \longrightarrow & \mathcal{K} \end{array}$$

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(A similar result holds for Moore: replace the comma $((- \otimes I)/O)$ with the slice $\mathcal{K}/O$.)

X-automata

(cf. Adámek-Trnková)

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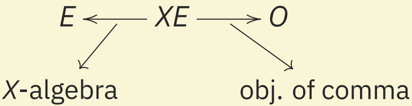

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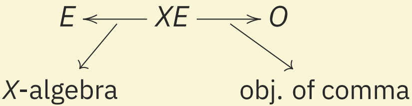

which yields at once a pullback characterization of X-automata, ...

X-automata

X-**Mly** fits in a strict 2-pullback

$$
\begin{array}{ccc}
X\text{-}\mathbf{Mly} & \xrightarrow{U'} & (X/O) \\
{\scriptstyle V'}\downarrow & \lrcorner & \downarrow{\scriptstyle V} \\
\mathbf{Alg}(X) & \xrightarrow[U]{} & \mathcal{K}
\end{array}
$$

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...and in particular when $X \dashv R$

Theorem
*The category of X-automata is cocomplete when $\mathcal{K}$ is, with colimits created by a canonical functor X-**Mly** $\to \mathcal{K}$; it is complete when $\mathcal{K}$ is.*

X-automata

From [Mac Lane, V.6, Ex. 3]: in every strict pullback of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U'} & \mathcal{B} \\ {\scriptstyle V'}\downarrow & \lrcorner & \downarrow{\scriptstyle V} \\ \mathcal{C} & \xrightarrow[U]{} & \mathcal{K} \end{array}$$

if U creates, and V preserves, limits of a given shape $\mathcal{J}$, then U' creates limits of shape $\mathcal{J}$.

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But $\mathbf{Alg}(X) \to \mathcal{K}$ creates all limits, and $X/O \to \mathcal{K}$ all connected limits; thus the problem boils down to find a terminal object (products follow).

X-automata

Claim: the terminal object of X-**Mly** is the terminal coalgebra for the functor

$$A \mapsto RA \times O$$

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$$d_\infty : X\left(\prod_{n \geqslant 1} R^n O \right) \to \prod_{n \geqslant 1} R^n O$$
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A similar line of reasoning leads to the terminal object in X-**Mre** being $O_\infty = \prod_{n \geqslant 0} R^n O$.

X-automata

How to induce a terminal morphism

$$\begin{array}{ccccc}
E & \xleftarrow{\;d\;} & XE & \xrightarrow{\;s\;} & O \\
\big\downarrow{\scriptstyle !_E} & & \big\downarrow{\scriptstyle X!_E} & & \big\| \\
O_\infty & \xleftarrow[d_\infty]{} & XO_\infty & \xrightarrow[s_\infty]{} & O
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$!_E : E \to O_\infty$ is defined as

$$d_\infty : \mathrm{mate}\big(\cdots \to XXXE \xrightarrow{XXd} XXE \xrightarrow{Xd} XE \xrightarrow{s} O \big)$$

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Products are computed as pullbacks along terminal maps:

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P_\infty & \longrightarrow & F \\
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- **Mly**(I, O) complete with terminal object $[I^+, O]$ ($\leftarrow$ free semigroup).
- **Mre**(I, O) complete with terminal object $[I^*, O]$ ($\leftarrow$ free monoid).

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There is a composite of adjoint functors

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where B is a 'behaviour' functor defined as

$$(E, d, s) \mapsto (!_E : E \to O_\infty)$$

and its left adjoint L is determined through the 'free' Moore machine on a X-algebra over the terminal O_∞.

Bicategories

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Bicategories

So far so good.

It's all fun and games until someone ~~loses an eye~~uses a bicategory.

- under which assumptions is **Mly**(I, O) the hom-category of a bicategory?
- similar question for Mealy.
- A monoidal category is just™ a bicategory with a single object; but then what is a Mealy automaton in a bicategory $\mathbb{B}$?

Bicategories of Automata

The bicategory $\underline{\mathbf{Mly}}$

Let $\mathcal{K}$ be a Cartesian category. Define a bicategory $\underline{\mathbf{Mly}}_{\mathcal{K}}$ as follows

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7. the associator and the unitors are inherited from $\mathcal{K}$.

Composition of 1-cells

Given two Mealy machines $(E, d, s) : I \to J$ and $(F, d', s') : J \to K$,

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where $E \xleftarrow{d} E \times I \xleftarrow{\pi_F} E \times F \times I \xrightarrow{E \times s} F \times J \xrightarrow{d'} F$

Composition of 1-cells

Proof of associativity is bureaucracy:

$$
\begin{aligned}
(d_1 \diamondsuit d_2) \diamondsuit d_3 &= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1, d_1 \cdot (E_1 \times s_2)\rangle \cdot (E_1 \times E_2 \times s_3)\rangle \\
&= \langle d_3 \cdot \pi_{12}, \langle d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3), d_1 \cdot (E_1 \times s_2) \cdot (E_1 \times E_2 \times s_3)\rangle\rangle \\
&= \langle d_3 \cdot \pi_{12}, \underline{\langle d_2 \cdot \pi_1 \cdot (E_1 \times E_2 \times s_3)}, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle\rangle \\
d_1 \diamondsuit (d_2 \diamondsuit d_3) &= \langle\langle d_3 \cdot \pi_2, d_2 \cdot (E_2 \times s_3)\rangle \cdot \pi_1, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle \\
&= \langle\langle d_3 \cdot \pi_2 \cdot \pi_1, d_2 \cdot (E_2 \times s_3) \cdot \pi_1\rangle, d_1 \cdot (E_1 \times (s_2 \cdot (E_2 \times s_3)))\rangle \\
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\end{aligned}
$$

Unitality follows a similar (simpler) strategy.

Corollar(ies)

- there are bicategories
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- if $\mathcal{K}$ is Cartesian closed, all right/left Kan extensions/lifts exist;

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? the assignment $\mathcal{K} \mapsto \underline{\mathbf{Mly}}_{\mathcal{K}}$ is (2-)functorial $\mathbf{CCat} \to 2\text{-}\mathbf{Cat}$ (careful with the 2-cells, Eugene).

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? Guitart defines a 'bicategory of Mealy machines' as $\mathbf{Spn}_F(\mathbf{Mon})$, spans in **Cat** between monoids whose left leg is a fibration. Interesting adjunctions with our $\underline{\mathbf{Mly}}$'s?

Automata in bicategories

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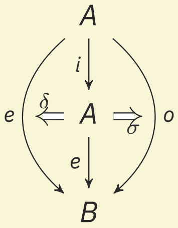
A monoidal category is just a bicategory with a single object. What is a machine inside a bicategory $\mathbb{B}$ with objects $A, B, C \ldots$? A bicategorical Moore machine consists of a span of 2-cells in $\mathbb{B}$

$$e \xLeftarrow{\delta} e \circ i \xRightarrow{\sigma} o$$

or rather a diagram of 2-cells



for objects $A, B \in \mathbb{B}$.

Examples

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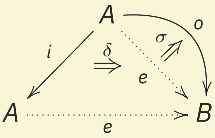
- so, iterated compositions $i \circ \cdots \circ i$ make sense as much as iterated tensor powers $I \otimes \cdots \otimes I$ made sense in $\mathcal{K}$;
- one can find examples in
  - categories, functors and natural transformations;
  - categories, functors and lax transformations;
  - categories, profunctors and 2-cells (a fortiori, in **Rel**);
  - sets and metric relations;
  - topological, approach, closure spaces,...

Behaviour as a Kan extension

A bicategorical Moore machine in $\mathbb{B}$ of fixed input and output i, o is a diagram



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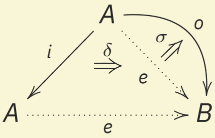

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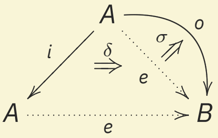

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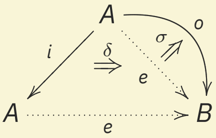

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- from the multiplication $\boldsymbol{\mu} : i^\natural \circ i^\natural \Rightarrow i^\natural$ get $\mathrm{Ran}_{i^\natural} \Rightarrow \mathrm{Ran}_{i^\natural} \circ \mathrm{Ran}_{i^\natural}$, and thus

$$\delta : \mathrm{Ran}_{i^\natural} o \circ i \xrightarrow{\mathrm{Ran}_{i^\natural} o * \boldsymbol{\eta}} \mathrm{Ran}_{i^\natural} o \circ i^\natural \Longrightarrow \mathrm{Ran}_{i^\natural} o$$

Intertwiners

Definition

An intertwiner $(u, v) : (e, \delta, \sigma)_{A,B} \looparrowright (e', \delta', \sigma')_{A',B'}$ consists of a pair of 1-cells $u : A \to A', v : B \to B'$ and a triple of 2-cells ι, ϵ, ω disposed as follows:

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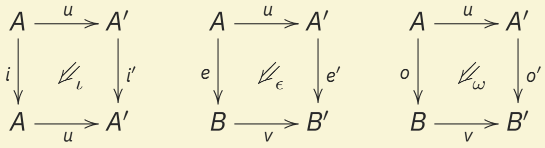

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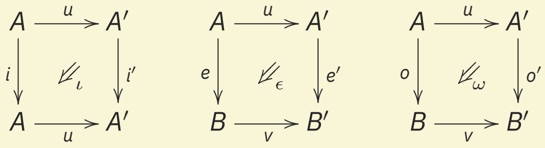

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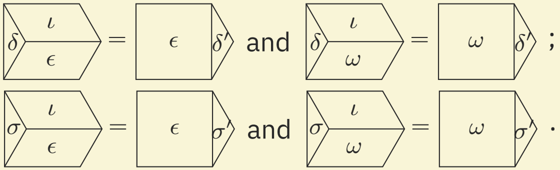

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Let $(u, v), (u', v') : (e, \delta, \sigma) \nrightarrow (e', \delta', \sigma')$ be two parallel intertwiners between bicategorical Mealy machines; a 2-cell $(\varphi, \psi) : (u, v) \Rightarrow (u', v')$ consists of a pair of 2-cells $\varphi : u \Rightarrow u'$, $\psi : v \Rightarrow v'$ such that the following identities hold true:

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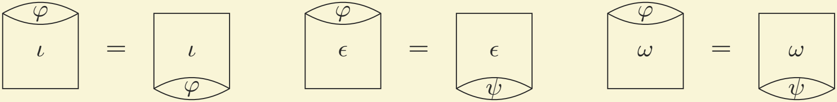

Intertwiners

Specialized to the monoidal case, the previous two definitions become

- **morphisms** of type $\iota : I' \otimes U \to V \otimes I, \epsilon : E' \otimes U \to V \otimes E, \omega : O' \otimes U \to V \otimes O;$

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- **pairs** $f : U \to U'$ and $g : V \to V'$ such that

$$\begin{array}{ccc} E' \otimes I' \otimes U & \xrightarrow{d' \otimes U} & E' \otimes U \\ {\scriptstyle E' \otimes I' \otimes f}\downarrow & & \downarrow{\scriptstyle E' \otimes f} \\ E' \otimes I' \otimes U' & \xrightarrow[d' \otimes U']{} & E' \otimes U' \end{array} \qquad \begin{array}{ccc} V \otimes E \otimes I & \xrightarrow{V \otimes d} & V \otimes E \\ {\scriptstyle g \otimes E \otimes I}\downarrow & & \downarrow{\scriptstyle g \otimes E} \\ V' \otimes E \otimes I & \xrightarrow[V' \otimes d]{} & V' \otimes E \end{array}$$

Open problems

Of other bicategories

In 1974 Guitart defined a bicategory of Mealy machines:

- the objects are categories $M, N, \dots$ (actually, monoids);
- the 1-cells are spans

$$M \xleftarrow{D} \mathcal{E} \xrightarrow{S} N$$

  where D is a fibration and S is a functor.
- composition of 1-cells is as in **Span**.
- G then proves that **MAC** is the Kleisli bicategory of the diagram monad $C \mapsto \mathbf{Cat}/\!\!/C$;

We conjecture the existence of a left pseudo-adjoint L in

$$L : \underline{\mathbf{Mly}_{\mathbf{Cat}}} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{MAC} : G$$

Nondeterminism in equipments

- In **Rel**, $R = \mathrm{Ran}_{I^\natural} O$ is the relation defined as

$$(a, b) \in R \iff \forall a' \in A.((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

  This relation expresses *reachability* of b from a:

$$a\, R\, b \iff \Big((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a'\, O\, b\Big)$$

- Passing from automata in **Cat** to automata in Prof accounts for a form of nondeterminism; one can conjecture to be able to address *nondeterministic* BA in $\mathbb{B}$ as *deterministic* BA in a proarrow equipment.

Thanks!

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Get in touch if you have ideas / want to join!