The couniversal property of the Chu construction

October 9, 2020

Fact
The Chu construction of an autonomous (=symmetric monoidal closed) category $(\mathcal{A}, \mathbf{d})$ is the comma category $(\mathcal{A}/(-)^{\mathbf{d}})$ of the functor

$$(-)^{\mathbf{d}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{A} : A \mapsto (A \multimap \mathbf{d})$$

Chu$(\mathcal{A}, \mathbf{d})$ is *-autonomous, i.e. homming into **d** is a (contravariant) involution, and universally so.

This gives the Chu construction a universal property in the category of object-preserving functor; in fact, Chu is the cofree *-autonomous category on $(\mathcal{A}, \mathbf{d})$

Remark
There exist an adjoint triple

$$\begin{array}{ccc} & \xleftarrow{\qquad c \qquad} & \\ \mathcal{C} & \xrightarrow[\quad \bot \quad]{\quad i \ \bot \quad} & \mathcal{C}^{\to} \\ & \xleftarrow[\qquad d \qquad]{} & \end{array}$$

where i is fully faithful and ambimonadic (monadic and comonadic); there is an equivalence between the $i.c$-algebras and the $i.d$-coalgebras.

A morphism if a $i.c$-algebra if and only if it is invertible (and the algebra structure provides its inverse).

Looking for a laxified version of this result, we shall replace the adjunction $i \dashv d$ with an adjunction

$$\mathsf{Cat} \underset{R}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Cat}^{\rightarrow}_{lax}$$

where the typical morphism $\begin{bmatrix} \mathcal{A} \\ F\downarrow \\ \mathcal{C} \end{bmatrix} \overset{(u,v)}{\longrightarrow} \begin{bmatrix} \mathcal{B} \\ G\downarrow \\ \mathcal{D} \end{bmatrix}$ of $\mathsf{Cat}^{\rightarrow}_{lax}$ is a lax-commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{u} & \mathcal{B} \\ F\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow G \\ \mathcal{C} & \xrightarrow[v]{} & \mathcal{D} \end{array}$$

$$\mathrm{Cat} \underset{R}{\overset{I}{\rightleftarrows}} \mathrm{Cat}^{\to}_{lax} \quad (I \dashv R)$$

The functor I is a “degeneration” that sends a category to its identity arrow, and acts on morphisms accordingly (the square commutes strictly).

Its right adjoint functor R is a “lax face” that sends a functor $F : \mathcal{A} \to \mathcal{B}$ to the comma category $(F/\mathcal{B})$.

$$\begin{array}{ccc} (F/\mathcal{B}) & \longrightarrow & \mathcal{A} \\ \big\downarrow & \swArrow & \big\downarrow F \\ \mathcal{B} & = & \mathcal{B} \end{array}$$

R splits into a composition

$$\mathsf{Cat}^{\rightarrow}_{lax} \xrightarrow{\quad\Delta\quad} \mathsf{Cat}^{\rightarrow}_{lax} \times_t \mathsf{Cat}^{\rightarrow}_{lax} \xrightarrow{\;(-/d_1-)\;} \mathsf{Cat}$$

where Δ is the diagonal functor, $\mathsf{Cat}^{\rightarrow}_{lax} \times_t \mathsf{Cat}^{\rightarrow}_{lax}$ is the category obtained as the strict pullback

$$\begin{array}{ccc} \mathsf{Cat}^{\rightarrow}_{lax} \times_t \mathsf{Cat}^{\rightarrow}_{lax} & \longrightarrow & \mathsf{Cat}^{\rightarrow}_{lax} \times \mathsf{Cat}^{\rightarrow}_{lax} \\ \big\downarrow \quad \lrcorner & & \big\downarrow (d_1, d_1) \\ \mathsf{Cat} & \longrightarrow & \mathsf{Cat} \times \mathsf{Cat} \end{array}$$

and $(-/d_1-) : \mathsf{Cat}^{\rightarrow}_{lax} \times_t \mathsf{Cat}^{\rightarrow}_{lax} \to \mathsf{Cat}$ is the comma functor that sends F into (F/d_1F).

In the adjunction

$$\mathrm{Cat} \underset{R}{\overset{I}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Cat}^{\rightarrow}_{lax}$$

the functor I is 2-fully faithful, and the category of coalgebras for the comonad $P = I.R$ coincides with the subcategory of $\mathrm{Cat}^{\rightarrow}_{lax}$ spanned by the equivalence of categories.

Unwinding the structure of a P-coalgebra on a functor $F : \mathcal{A} \to \mathcal{B}$: it amounts to 2-cells

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{s} & (F/\mathcal{B}) \\ {\scriptstyle F}\big\downarrow & \swarrow_{k} & \Big\| \\ \mathcal{B} & \xrightarrow[t]{} & (F/\mathcal{B}) \end{array}$$

such that (unit condition)

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{s} & (F/\mathcal{B}) & \xrightarrow{p_0} & \mathcal{A} \\ \big\downarrow & \swarrow_{k} & \Big\| & \swarrow & \big\downarrow \\ \mathcal{B} & \xrightarrow[t]{} & (F/\mathcal{B}) & \xrightarrow[p_1]{} & \mathcal{B} \end{array}$$

is the identity cell of F; this entails $\eta : p_0 s = 1 \Rightarrow p_0 t F = t_0 F$ is an isomorphism; a similar condition on the (undrawn) comultiplication condition entails that $\epsilon : F t_0 \Rightarrow 1$ is an isomorphism; thus, $F \dashv t_0$ is an equivalence of categories.

Fact
A pair of functors $F : \mathcal{A} \leftrightarrows \mathcal{B} : G$ are adjoint if and only if there is a fibered equivalence of commæ

$$\begin{array}{ccc} (F/\mathcal{B}) & \xrightarrow{\sim} & (\mathcal{A}/G) \\ & \searrow \quad \swarrow & \\ & \mathcal{A} \times \mathcal{B} & \end{array}$$

Corollary: an endofunctor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ is self-adjoint if and only if the comma construction provides a fibered contravariant equivalence (“fibered duality”)

$$(F/\mathcal{A}^{\mathrm{op}}) \cong (F/\mathcal{A}^{\mathrm{op}})^{\mathrm{op}}$$

Consider the category SAdj $\subset \mathrm{Cat}^{\rightarrow}_{lax}$ spanned by the endofunctors $F : \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ that are self-adjoint, meaning that $F \dashv F^{\mathrm{op}}$. We fix notation as follows:

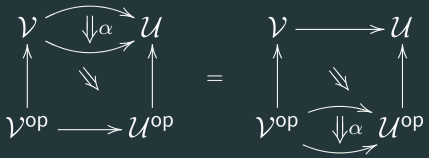
- $(-)^{\mathbf{d}} : \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ is a self-adjoint endofunctor, $A^{\mathbf{d}}$ its action on objects;
- the unit $\eta : A \to A^{\mathbf{dd}}$, such that

$$\mathcal{V}(A, B^{\mathbf{d}}) \cong \mathcal{V}(B, A^{\mathbf{d}})$$

- morphisms in SAdj are lax-commutative squares

$$\begin{array}{ccc}
\mathcal{V}^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} & \mathcal{U}^{\mathrm{op}} \\
{\scriptstyle (-)^{\mathbf{d}}_{\mathcal{V}}}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle (-)^{\mathbf{d}}_{\mathcal{U}}} \\
\mathcal{V} & \xrightarrow[F]{} & \mathcal{U}
\end{array}$$

- A 2-cell is an invertible natural transformation $\alpha : F \Rightarrow G$ such that

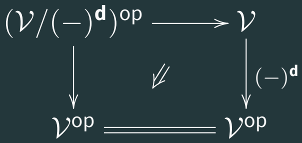


All together, this hints how to build a right adjoint to the functor

$$\text{SAdj} \longleftarrow \text{Dual} : i$$

defined on objects to be the comma construction

$$(\mathcal{V}, (-)^{\mathbf{d}}) \mapsto (\mathcal{V}/(-)^{\mathbf{d}})^{\text{op}}$$



The category $(\mathcal{V}/(-)^{\mathbf{d}})^{\mathrm{op}}$ has an obvious "swap" self-equivalence

$$\begin{bmatrix} A \\ f\downarrow \\ B^{\mathbf{d}} \end{bmatrix} \mapsto \begin{bmatrix} B \\ \downarrow \\ A^{\mathbf{d}} \end{bmatrix}$$

sending every arrow to its mate. By construction it is an equivalence.

This acts in the obvious way on 1- and 2-cells of SAdj; this defines a functor

$$\mathbb{C} : \mathsf{Dual} \leftarrow \mathsf{SAdj}$$

right adjoint to *i* above; the category of coalgebras for the comonad $i\mathbb{C}$ is (equivalent to) Dual, so *i* is 2-comonadic.

This construction specialises to the case where the self-adjunction is given by homming with a prescribed object **d** of an autonomous $\mathcal{A}$:

- The category $\mathrm{Aut}_{\mathbf{d}}$ has objects the pointed autonomous categories where the self-adjunction is given by homming with the distinguished object **d**:

  $$(-)^{\mathbf{d}} : \mathcal{V}^{\mathrm{op}} \to \mathcal{V} : A \mapsto (A \multimap \mathbf{d})$$

  We also require finite limits and colimits in $\mathcal{V}$.

A functor of pointed autonomous categories is a lax monoidal pointed functor, colax closed, and colax preserving the distinguished object.

There is a diagram

$$\begin{array}{ccc}
*\mathrm{Aut}_{\mathbf{d}} & \underset{\dashleftarrow}{\overset{U}{\longrightarrow}} \; \bot & \mathrm{Aut}_{\mathbf{d}} \\
\downarrow & & \downarrow \\
\mathrm{Dual} & \underset{\mathbb{C}}{\overset{i}{\rightleftarrows}} \; \bot & \mathrm{SAdj}
\end{array}$$

that can be filled with a right adjoint to $U : *\mathrm{Aut}_{\mathbf{d}} \to \mathrm{Aut}_{\mathbf{d}}$, yielding the cofree *-autonomous category on an autonomous category $\mathcal{A}$.

$$\underline{\mathsf{Chu}} : \mathsf{Aut}_{\mathbf{d}} \longrightarrow *\mathsf{Aut}_{\mathbf{d}}$$

The functor $\underline{\mathsf{Chu}}$ acts on pointed autonomous categories sending $(\mathcal{A}, (- \multimap \mathbf{d}))$ into the comma category $(\mathcal{A}/(- \multimap \mathbf{d}))$; the closed structure of $\underline{\mathsf{Chu}}(\mathcal{A}, \mathbf{d})$ can entirely be deduced from the autonomous structure of $(\mathcal{A}, \mathbf{d})$.

The tensor structure can in particular be deduced from the internal hom, because the duality $(- \multimap \mathbf{d}) : \mathcal{A}^{\mathrm{op}} \to \mathcal{A}$ imposes that

$$\begin{bmatrix} A \\ f \downarrow \\ B^{\mathbf{d}} \end{bmatrix} \otimes_{\underline{\mathsf{Chu}}} \begin{bmatrix} C \\ g \downarrow \\ D^{\mathbf{d}} \end{bmatrix} := \begin{bmatrix} A \\ f \downarrow \\ B^{\mathbf{d}} \end{bmatrix}^{*} \multimap_{\underline{\mathsf{Chu}}} \begin{bmatrix} C \\ g \downarrow \\ D^{\mathbf{d}} \end{bmatrix}$$

To define $\multimap_{\text{Chu}}$, recall that the self-adjunction in $(\mathcal{A}, \mathbf{d})$ that we want to turn into an equivalence is the "homming with $\mathbf{d}$" functor $(-)^{\mathbf{d}} = - \multimap \mathbf{d}$: such functor defines maps $\hat{k} : (D \multimap B) \to (B^{\mathbf{d}} \multimap D^{\mathbf{d}})$, obtained as mates of the composition.

Given $X = \begin{bmatrix} A \\ f \downarrow \\ B^{\mathbf{d}} \end{bmatrix}$ and $Y = \begin{bmatrix} C \\ g \downarrow \\ D^{\mathbf{d}} \end{bmatrix}$ the object $X \multimap Y$ is defined through the pullback

$$\begin{array}{ccc}
P & \longrightarrow & D \multimap B \\
\big\downarrow & \lrcorner & \big\downarrow \hat{k} \\
 & & B^{\mathbf{d}} \multimap D^{\mathbf{d}} \\
\big\downarrow & & \big\downarrow f \multimap D^{\mathbf{d}} \\
A \multimap C & \xrightarrow[a \multimap g]{} & A \multimap D^{\mathbf{d}}
\end{array}$$

As a direct consequence, $X \otimes_{\underline{\mathsf{Chu}}} Y = X^* \multimap_{\underline{\mathsf{Chu}}} Y$ is defined as $\begin{bmatrix} B \\ \bar{f}\downarrow \\ A^{\mathbf{d}} \end{bmatrix} \multimap_{\underline{\mathsf{Chu}}} \begin{bmatrix} C \\ g\downarrow \\ D^{\mathbf{d}} \end{bmatrix}$, i.e. by the pullback

$$
\begin{array}{ccccc}
Q & \longrightarrow & & & D \multimap A \\
\downarrow & \lrcorner & & & \downarrow \\
 & & & & D \multimap B^{\mathbf{d}} \\
 & & & & \| \\
B \multimap C & \longrightarrow & B \multimap D^{\mathbf{d}} & = & (B \otimes D)^{\mathbf{d}}
\end{array}
$$

«which is actually simpler to memorise than $X \multimap_{\underline{\mathsf{Chu}}} Y$, but not as directly motivated».

- it follows that

$$X \multimap_{\underline{\mathsf{Chu}}} Y \cong (X^* \multimap_{\underline{\mathsf{Chu}}} Y^*)^* \qquad \begin{bmatrix} A \\ f\downarrow \\ B^{\mathbf{d}} \end{bmatrix}^* = \begin{bmatrix} A \\ f\downarrow \\ B^{\mathbf{d}} \end{bmatrix} \multimap_{\underline{\mathsf{Chu}}} \mathbf{d}$$

To define the arrow part of $\underline{\text{Chu}}$, let $F : \mathcal{V} \to \mathcal{U}$ be a functor such that

$$\begin{array}{ccc} \mathcal{V}^{\text{op}} & \xrightarrow{F^{\text{op}}} & \mathcal{U}^{\text{op}} \\ {\scriptstyle (-)^{\mathbf{d}}_{\mathcal{V}}} \big\downarrow & \overset{\varphi}{\nearrow\!\!\!\nearrow} & \big\downarrow {\scriptstyle (-)^{\mathbf{d}}_{\mathcal{U}}} \\ \mathcal{V} & \xrightarrow[F]{} & \mathcal{U} \end{array}$$

Let $FX = F \left[\begin{array}{c} A \\ f \downarrow \\ B^{\mathbf{d}} \end{array} \right] := FA \xrightarrow{Ff} F(B^{\mathbf{d}}) \xrightarrow{\varphi} (FB)^{\mathbf{d}}$.

From this,

$$\begin{array}{ccccccc} FB & \longrightarrow & F(B^{\mathbf{dd}}) & \longrightarrow & F(A^{\mathbf{d}}) & \longrightarrow & (FA)^{\mathbf{d}} \\ \| & & & \searrow & & & \| \\ FB & \longrightarrow & (FB)^{\mathbf{dd}} & \longrightarrow & F(B^{\mathbf{d}})^{\mathbf{d}} & \longrightarrow & (FA)^{\mathbf{d}} \end{array}$$

is a morphism $F(X^*) \to (FX)^*$.

The map above is a canonical morphism
$F(X \multimap_{\underline{\mathrm{Chu}}} \mathbf{d}) \to FX \multimap_{\underline{\mathrm{Chu}}} \pm$ that renders F colax closed.

$\pm := \left[\begin{array}{cc} & I \\ \eta_I & \downarrow \\ & I^{\mathbf{dd}} \end{array}\right]$ is the monoidal unit of $\underline{\mathrm{Chu}}(\mathcal{A}, \mathbf{d})$, and the dualising object.

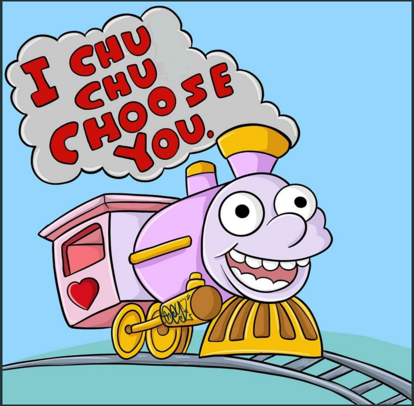
The functor $\underline{\text{Chu}} : \text{Aut}_{\mathbf{d}} \to *\text{Aut}_{\mathbf{d}}$ is right adjoint to the embdding $U : *\text{Aut}_{\mathbf{d}} \to \text{Aut}_{\mathbf{d}}$.

U is comonadic: coalgebras and (pseudo-)homomorphisms correspond to *-autonomous categories and their morphisms.

With more work, a similar result could be proved for Hyland–de Paiva's *Dialectica* categories.

Just like $\underline{\text{Chu}}$, Dial can be reduced to a comma construction, only this time further enriched (over posets).

Lax coalgebras for the Dial comonad are precisely Dialectica categories.



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