Differential 2-rigs

February 8, 2021

Motivations

Galois theory asserts that to each polynomial equation one can attach a finite group, the Galois group of the polynomial, in such a way that the roots of a polynomial can be found via algebraic operations and root extraction if and only if its Galois group is solvable.

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A group G is solvable if there exists a chain of subgroups $1 \leq G_1 \leq \cdots \leq G_n \leq G$ with the property that each G_i is normal in G_{i+1}, and each quotient G_{i+1}/G_i is abelian.

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Turns out that a Galois correspondence for differential equations is possible, paying a price:

- the group attached to a differential equation is not finite any more;
- it has a non-trivial topology, and the "correct" subgroups to consider are the closed ones;
- more than often, such groups are algebraic manifolds of infinite dimension.

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The theory of reductive algebraic groups arose as a way to understand the operation of adding a solution to $y' = y$ to the ring of polynomials. (=exponential elements; they live in rings of power series).

A differential algebra over k is a k-algebra R endowed with a endomorphism $d : R \to R$ that is k-linear, and satisfies the Leibniz rule:

$$d(a.b) = da.b + a.db$$

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A differential equation in R is an equation of the form $F(y, y^{(1)}, y^{(2)}, \dots) = 0$ where F is a polynomial with coefficients in R, and $y^{(1)} := dy, y^{(n)} = d(y^{(n-1)})$.

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- Ring operations
- addition of integrals (=solutions to $y' = a, a \in R$)
- addition of exponentials (=solutions to $y' = by, b \in R$).

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A notoriously non-solvable group is that of the diff. eq'n $y = e^{-x^2}$.

Slogan
Abstracting the theory of differential rings is a secure path to unexpectedly profound mathematics.

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Guided by this, the plan is to transform rings into (rig) categories $\mathcal{R}$ and endomorphisms $d : R \to R$ into functors $\partial : \mathcal{R} \to \mathcal{R}$, and the Leibniz condition

$$\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$$

into an isomorphism natural in A, B.

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Rings

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In simple terms, we want to capture a notion that categorifies rigs (rings without additive inverses).

A 2-rig is a particular instance of a more general notion, first introduced by Laplaza: a category $\mathcal{C}$ with two monoidal structures $\otimes, \oplus$, such that '$\otimes$ distributes over $\oplus$'.

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As natural as his axiomatics may seem, the precise formalisation of a coherence theorem for a distributive category requires a lot of effort and numerous diagrams: see Laplaza for the precise definition, revolving around 'distributor maps'

$$\left\{ \delta^{\mathsf{R}} : (A \cup B) \otimes C \to A \otimes C \cup B \otimes C \delta^{\mathsf{L}} : A \otimes (B \cup C) \to A \otimes B \cup A \otimes C \right.$$

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- A commutative 2-rig is a category $\mathcal{C}$, where $\otimes$ is also symmetric;

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Combinations are possible: it is clear what a commutative closed 2-rig is.

The following are examples of 2-rigs:

- The category $(\mathsf{Set}, \times, 1)$ of sets and functions is a commutative closed 2-rig; more generally, all cartesian closed categories with coproducts $(\mathcal{A}, \times, 1)$ are bicartesian categories.

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- The category $(\mathsf{Mod}_R, \otimes, R)$ of modules over a ring R is a commutative closed 2-rig.

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- The category $(\mathrm{Set}, \times, 1)$ of sets and functions is a commutative closed 2-rig; more generally, all cartesian closed categories with coproducts $(\mathcal{A}, \times, 1)$ are bicartesian categories.
- The category $(\mathrm{Mod}_R, \otimes, R)$ of modules over a ring R is a commutative closed 2-rig.
- The category of (real or complex) topological vector bundles over a topological space X, equipped with the tensor product of vector bundles is a 2-rig (where $\cup$ is the direct sum of vector bundles taking the bundle associated with fiberwise Vect-coproduct).

- Given a monoidal category $(\mathcal{A}, \oplus, j)$ the category $([\mathcal{A}^{\mathrm{op}}, \mathrm{Set}], *, yj = \mathcal{A}(j, -))$ of presheaves over $\mathcal{A}$ endowed with the Day convolution monoidal structure

$$F * G := \int^{U,V \in \mathcal{A}} FU \times GV \times \mathcal{A}(U \oplus V, -)$$

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Note that $[\mathcal{A}^{\mathrm{op}}, \mathrm{Set}]$ is closed no matter what $\oplus$ is, and the internal hom can be computed as

$$\{G, H\} : A \mapsto \int_X \mathrm{Set}(GX, H(A \oplus X))$$

Derivations

Definition (Derivation on a 2-rig)

A derivation on a 2-rig is a functor $\partial : \mathcal{C} \to \mathcal{C}$ having the following properties:

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\begin{array}{ccccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{(\partial \otimes \mathcal{C}, \mathcal{C} \otimes \partial)} & \mathcal{C} \times \mathcal{C} \\
{\scriptstyle \otimes}\downarrow & & \iota \Uparrow & & \downarrow{\scriptstyle \cup} \\
\mathcal{C} & & \xrightarrow[\partial]{\hspace{8em}} & & \mathcal{C}
\end{array}
$$

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where $\Delta_{\mathcal{C} \times \mathcal{C}}$ is the diagonal functor $(A, B) \mapsto (A, B, A, B)$

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{\scriptstyle \otimes}\downarrow & & \wr \Uparrow & & \downarrow{\scriptstyle \cup} \\
\mathcal{C} & & \xrightarrow[\partial]{\qquad\qquad} & & \mathcal{C}
\end{array}
$$

where $\Delta_{\mathcal{C} \times \mathcal{C}}$ is the diagonal functor $(A, B) \mapsto (A, B, A, B)$ and $(\partial \otimes \mathcal{C}, \mathcal{C} \otimes \partial)$ does the obvious thing.

The cell $\mathfrak{l}$ is called the leibnizator, has components $\mathfrak{l}_{AB} : \partial(A \otimes B) \Rightarrow \partial A \otimes B \cup A \otimes \partial B$, and it is subject to a bunch of coherence conditions:

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Compatibility with the right distributor:

$$
\begin{array}{ccc}
\partial((Y \cup Z) \otimes X) & \xrightarrow{\delta^{\mathsf{R}}} & \partial(Y \otimes X \cup Z \otimes X) \\
\Big\downarrow \wr & & \Big\downarrow \wr \\
(Y \cup Z) \otimes \partial X \cup \partial(Y \cup Z) \otimes X & & \partial(Y \otimes X) \cup \partial(Z \otimes X) \\
\Big\downarrow \delta^{\mathsf{R}} \cup \delta^{\mathsf{R}} & & \Big\downarrow \wr \cup \wr \\
Y \otimes \partial X \cup Z \otimes \partial X \cup \partial Y \otimes X \cup \partial Z \otimes X & \xrightarrow[\sim]{} & \partial Y \otimes X \cup Y \otimes \partial X \cup \partial Z \otimes X \cup Z \otimes \partial X
\end{array}
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\wr\downarrow & & \downarrow\wr \\
(Y \cup Z) \otimes \partial X \cup \partial(Y \cup Z) \otimes X & & \partial(Y \otimes X) \cup \partial(Z \otimes X) \\
{\scriptstyle \delta^{\mathsf{R}} \cup \delta^{\mathsf{R}}}\downarrow & & \downarrow{\scriptstyle \wr \cup \wr} \\
Y \otimes \partial X \cup Z \otimes \partial X \cup \partial Y \otimes X \cup \partial Z \otimes X & \xrightarrow[\sim]{} & \partial Y \otimes X \cup Y \otimes \partial X \cup \partial Z \otimes X \cup Z \otimes \partial X
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(Y \cup Z) \otimes \partial X \cup \partial(Y \cup Z) \otimes X & & \partial(Y \otimes X) \cup \partial(Z \otimes X) \\
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Y \otimes \partial X \cup Z \otimes \partial X \cup \partial Y \otimes X \cup \partial Z \otimes X & \xrightarrow[\sim]{} & \partial Y \otimes X \cup Y \otimes \partial X \cup \partial Z \otimes X \cup Z \otimes \partial X
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\downarrow \wr & & \downarrow \wr \\
(Y\cup Z)\otimes\partial X\cup\partial(Y\cup Z)\otimes X & & \partial(Y\otimes X)\cup\partial(Z\otimes X)\\
\downarrow {\scriptstyle \delta^{\mathsf{R}}\cup\delta^{\mathsf{R}}} & & \downarrow {\scriptstyle \wr\cup\wr}\\
Y\otimes\partial X\cup Z\otimes\partial X\cup\partial Y\otimes X\cup\partial Z\otimes X & \xrightarrow[\sim]{} & \partial Y\otimes X\cup Y\otimes\partial X\cup\partial Z\otimes X\cup Z\otimes\partial X
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(Y \cup Z) \otimes \partial X \cup \partial(Y \cup Z) \otimes X & & \partial(Y \otimes X) \cup \partial(Z \otimes X) \\
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\end{array}
$$

where the unnamed isomorphisms are symmetries of $\cup$ or arising from the strong $\cup$-monoidality of ∂;

Compatibility with the left distributor:

$$
\begin{array}{ccc}
\partial(X \otimes (Y \cup Z)) & \xrightarrow{\delta^{\mathsf{L}}} & \partial(X \otimes Z \cup Y \otimes Z) \\
\downarrow{\scriptstyle \wr} & & \downarrow \\
\partial X \otimes (Y \cup Z) \cup X \otimes \partial(Y \cup Z) & & \partial(X \otimes Y) \cup \partial(X \otimes Z) \\
\downarrow{\scriptstyle \delta^{\mathsf{L}} \cup \delta^{\mathsf{L}}} & & \downarrow{\scriptstyle \wr \cup \wr} \\
\partial X \otimes Y \cup \partial X \otimes Z \cup X \otimes \partial Y \cup X \otimes \partial Z & \longrightarrow & \partial X \otimes Y \cup X \otimes \partial Y \cup \partial X \otimes Z \cup X \otimes \partial Z
\end{array}
$$

where the unnamed isomorphisms are symmetries of $\cup$ or arising from the strong $\cup$-monoidality of ∂;

Compatibility with the right and left annullator:

$$
\begin{array}{ccccc}
\partial(X \otimes 0) & & \xrightarrow{\sim} & & \partial 0 \\
\big\downarrow{\scriptstyle \mathfrak{l}_{X0}} & & & & \big\downarrow{\scriptstyle \wr} \\
\partial X \otimes 0 \cup X \otimes \partial 0 & \longrightarrow & 0 + 0 & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
\partial(0 \otimes X) & & \xrightarrow{\sim} & & \partial 0 \\
\big\downarrow{\scriptstyle \mathfrak{l}_{0X}} & & & & \big\downarrow{\scriptstyle \wr} \\
\partial 0 \otimes X \cup 0 \otimes \partial X & \longrightarrow & 0 + 0 & \longrightarrow & 0
\end{array}
$$

where the unnamed isomorphisms come from the fact that $A \otimes -$ and $- \otimes B$ preserve the initial object for all $A, B \in \mathcal{C}$;

Compatibility with the right and left $\otimes$-unitor:

$$\begin{tikzcd} \partial(X \otimes I) \arrow[r, "\partial\rho_X"] \arrow[d, "\mathfrak{l}_{XI}"'] & \partial X \arrow[ddl, bend left, "\iota_{\partial X}"] \\ \partial X \otimes I \cup X \otimes \partial I \arrow[d, "\rho_{\partial X} \cup X \otimes \partial I"'] & \\ \partial X \cup X \otimes \partial I & \end{tikzcd} \qquad \begin{tikzcd} \partial(I \otimes X) \arrow[r, "\partial\lambda_X"] \arrow[d, "\mathfrak{l}_{IX}"'] & \partial X \arrow[ddl, bend left, "\iota_{\partial X}"] \\ \partial I \otimes X \cup I \otimes \partial X \arrow[d, "\partial I \otimes X \cup \lambda_{\partial X}"'] & \\ \partial I \otimes X \cup \partial X & \end{tikzcd}$$

(the properties of ∂I, where I is the $\otimes$-monoidal unit, are quite a subtle business; we will come back to this later; note in particular that no axiom entails that $\partial I = 0$.).

Compatibility with the associator:

$$
\begin{array}{ccc}
\partial((A \otimes B) \otimes C) & \xrightarrow{\partial\alpha} & \partial(A \otimes (B \otimes C)) \\
{\scriptstyle \iota_{A\otimes B,C}}\downarrow & & \downarrow{\scriptstyle \iota_{A,B\otimes C}} \\
\partial(A \otimes B) \otimes C \cup (A \otimes B) \otimes \partial C & & \partial A \otimes (B \otimes C) \cup A \otimes \partial(B \otimes C) \\
{\scriptstyle \iota_{A,B}\otimes C \cup (A\otimes B)\otimes\partial C}\searrow & & \swarrow \\
& \partial A \otimes B \otimes C \cup A \otimes \partial B \otimes C \cup A \otimes B \otimes \partial C &
\end{array}
$$

First remarks

Directly from these definitions we can easily see that

- The derivative of the initial object $\varnothing$ must be the initial object (because ∂ preserves the empty coproduct, or because of the Leibniz rule);

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- by induction on n,

$$\partial(A^{\otimes n}) \cong n \cdot A^{\otimes(n-1)} \otimes \partial A$$

where $n \cdot -$ sends an object $X \in \mathcal{C}$ to the n-fold coproduct $X \cup \cdots \cup X$.

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  where $n \cdot -$ sends an object $X \in \mathcal{C}$ to the n-fold coproduct $X \cup \cdots \cup X$.
- again by induction,

$$\partial^n(X \otimes Y) = \coprod_{k=0}^{n} \binom{n}{k} \cdot \partial^{n-k} X \otimes \partial^k Y$$

  where $\binom{n}{k}$ is the set of k-elements subsets of $\{1, \ldots, n\}$.

Examples

Edge cases

Any 2-rig $\mathcal{C}$, endowed with the trivial derivation $\mathcal{C} \to \mathcal{C}$ that is the constant functor at the initial object (regarded as empty coproduct).

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Let P be a distributive lattice; the identity functor $P \to P$ is, trivially, a derivation (because every element of P is $\vee$-idempotent).

Polynomials

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- objects are 'polynomials' $\sum_{i=0}^{d} A_i \otimes Y^i$, regarded as endofunctors $\mathcal{C} \to \mathcal{C}$, with the convention that $Y^0 = 1_{\mathcal{C}}$ and the action on an object X is given by

$$X \mapsto \sum_{i=0}^{d} A_i \otimes Y^i(X) \in \mathcal{C};$$

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- morphisms are natural transformations of functors.

The category $\mathcal{C}[Y]$ so obtained is a 2-rig where the sum is 'component-wise', and the $\otimes$-product is a similar 'Cauchy product' of polynomials.

Polynomials

let $\mathcal{C}$ be a 2-rig; $\mathcal{C}[Y]$ becomes a differential 2-rig if we endow $\mathcal{C}$ with the trivial derivation, and we put $\partial Y = I$, suitably extended on a generic expression $\sum_{i=0}^{d} A_i \otimes Y^i$ by linearity and Leibniz rule:

$$\partial\left(\sum_{i=0}^{d} A_i \otimes Y^i\right)) \cong \sum_{i=1}^{d} i \cdot A_i \otimes Y^{i-1}$$

let $\mathcal{C}$ be a differential 2-rig, with derivation denoted $a \mapsto \partial a$. One can define the 2-rig of differential polynomials with coefficients in $\mathcal{C}$ introducing an infinite set of 'variables' $\mathcal{Y} := \{Y, Y^{(1)}, Y^{(2)} \dots, Y^{(n)}, \dots\}$ and defining

$$\mathcal{C}[\mathcal{Y}] := \varinjlim \Big(\mathcal{C} \to \mathcal{C}[Y] \to \mathcal{C}[Y, Y^{(1)}] \to \mathcal{C}[Y, Y^{(1)}, Y^{(2)}] \to \dots\Big)$$

where we define inductively $\mathcal{C}[Y, Z] := \mathcal{C}[Y][Z]$, and the derivation as $\partial : Y^{(i)} \mapsto Y^{(i+1)}$.

A co-Heyting algebra is a bounded distributive lattice K such that $x \vee - : K \to K$ has a left adjoint $-\backslash x$ for all $x \in K$:

$$y\backslash x \leq z \iff y \leq x \vee z$$

Define $\partial x := x \wedge \lrcorner x$ it's easy to see that $\partial : K \to K$ is a derivation when K is regarded as a distributive 2-rig. Leibniz rule takes the form

$$\partial(a \wedge b) = (\partial a \wedge b) \vee (a \wedge \partial b).$$

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Example: lattice of subtoposes of a given topos $\mathcal{E}$; define the boundary $\partial\mathcal{A}$ of a subtopos $\mathcal{A} \subseteq \mathcal{E}$ in this lattice and then in turn the boundary ∂T of the geometric theory T that $\mathcal{A}$ classifies (see Caramello, 2009).

Results

Theorem

Let $\mathcal{C}$ be a 2-rig, and M a internal semigroup with multiplication $m : M \otimes M \to M$; then the map $\partial m : \partial M \otimes M \cup M \otimes \partial M \to \partial M$ splits as a pair of maps

$$\begin{cases} i_R : \partial M \otimes M \to \partial M \\ i_L : M \otimes \partial M \to \partial M \end{cases}$$

Then, i_R (resp., i_L) is a right (resp., left) action of M over ∂M.

- Let $\mathcal{C}$ be a 2-rig with a natural number object $\mathbf{N}$; then, $\mathbf{N}$ is a monoid in a canonical way, with respect to the morphism $\mathbf{N} \times \mathbf{N} \to \mathbf{N} : \lambda pq.s^p q$.

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  If $\mathcal{C}$ is a differential bicartesian category, then $\partial\mathbf{N}$ is a Lawvere dynamical system.
- Let $\mathcal{C}$ be an elementary topos, regarded as a bicartesian category; if $\mathcal{C}$ has a differential structure, the derivative $\partial\Omega$ of the subobject classifier is a module for the monoid $(\Omega, \wedge, \text{true})$.

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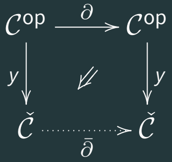
Then there exists a canonical extension $\bar{\partial}$ of ∂ to the additive presheaf category $\hat{\mathcal{C}} = [\mathcal{C}, Ab]$ (coproduct-linear and convolution-Leibniz), that hence becomes a differential 2-rig.



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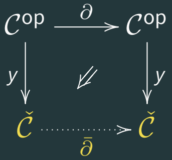

Recall the equivalence

$$[\mathsf{Fin}, \mathsf{Set}] \cong [\mathsf{Set}, \mathsf{Set}]_\omega$$

given by left Kan extension along the embedding $J : \mathsf{Fin} \to \mathsf{Set}$, where at the right hand side we put *finitary* endofunctors of Set.

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Given $F : \mathrm{Fin} \to \mathrm{Set}$, $T_F = Lan_J F$ is the associated finitary functor and there exists a unique monoidal structure $\diamond$ such that

$$Lan_J(F \diamond G) \cong Lan_J F \circ Lan_J G$$

Theorem (The chain rule)

Let $F, G : \mathsf{Fin} \to \mathsf{Set}$ *be non-*Σ*-species, and let* ∂ *be a derivation with respect to the cartesian monoidal structure; then,*

$$\begin{cases} \partial T_G(T_F A) \times \partial T_F A \cong \partial(T_G \circ T_F) \\ \partial(G \diamond F) = Lan_J \partial G(Fn) \times \partial Fn. \end{cases}$$

Questions

Vices and virtues of being $\partial 1$

In a differential ring, using the Leibniz rule: $d1 = d1 + d1$, which entails $d1 = 0$.

In a rig things are way more complicated: one has to postulate that $d1 = 0$, or derivation isn't even well-defined:

$$da = d(a \cdot 1) = da + a.d1 = da + \sum_{k=1}^{\infty} a.d1$$

Something similar happens in a differential 2-rig:

$$\partial I \cong \partial I \cup \partial I$$

from which we get the idea that ∂I is 'either empty or big' (e.g., in Set, ∂I is empty or -at least- countable.)

Vices and virtues of being $\partial 1$

Theorem
There is no nontrivial finite colimit-preserving derivation on the 2-rig $(\mathrm{Fin}, \times, 1)$ of finite sets and functions: such $\partial : \mathrm{Fin} \to \mathrm{Fin}$ is completely determined by its action on the point, so that

$$\partial A \cong \partial(A \cdot 1) \cong A \cdot \partial 1.$$

Same in the category of finite dimensional vector spaces, where $d = \dim V = 2d$ has 0 (thus the zero space) as unique solution.
Same in every category with a choice of dimension $\mathcal{C} \to \mathbb{N}$ for objects.

Vices and virtues of being $\partial 1$

Sometimes $\partial 1 = \partial 1 + \partial 1$ is forced to have just trivial solutions due to naturality;

Let ∂ be a derivation in a category of functors $\mathcal{A} \to \mathit{Set}$; then, $\partial 1$ is a functor $\mathcal{A} \to \mathit{Set}$ such that $F \cong F + F$, and naturally so; such functors must be constant on connected components of $\mathcal{A}$.

The unbearable largeness of $\partial 1$

Use Yoneda lemma.

Consider the hom-set $\hom(\partial 1, Z)$ for a generic object Z;

$\partial 1 \cong \partial 1 + \partial 1$ yields

$$\hom(\partial 1, Z) \cong \hom(\partial 1 + \partial 1, Z) \cong \hom(\partial 1, Z) \times \hom(\partial 1, Z).$$

So, $\hom(\partial 1, Z)$ can only be empty, a singleton or infinite.

Thus if a category is finite $\hom(\partial 1, Z)$ is either empty or a singleton, and in particular it must be a singleton when $Z = \partial 1$.

Coalgebras

If $\partial 1 \cong \partial 1 + \partial 1$, this means that $\partial 1$ is naturally a coalgebra for the ”leave it or double it” functor $S : A \mapsto A + A$, in such a way that there is a unique map

$$\begin{array}{ccc} \partial 1 & \cong & \partial 1 + \partial 1 \\ \downarrow & & \downarrow \\ C & \cong & C + C \end{array}$$

between $\partial 1$ and the terminal coalgebra of S; but wait, in the category of topological space C is the Cantor set! What just happened here?

Napier objects

As an endofunctor, ∂ might have interesting fixed points, and there is a standard procedure to build its initial algebra and terminal coalgebra.

Initial algebras are trivial, in that $\partial 0 = 0$ by using the Leibniz rule. On the other hand, the triviality of terminal coalgebras is governed by the shape of $\partial 1$:

$$1 \leftarrow \partial 1 \leftarrow \partial\partial 1 \leftarrow \partial\partial\partial 1 \leftarrow \dots$$

and the first ordinal λ for which the transition morphism $v : \partial^\lambda 1 \leftarrow \partial^{\lambda+1} 1$ is invertible realises the terminal coalgebra.

Derivatives of models

Let $(\mathcal{C}, J)$ be a *Grothendieck monoidal site* i.e. a Grothendieck topology such that the category of sheaves is monoidal; let $\partial : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ be a convolution-derivation; if ∂ is such that

a big black hole of ignorance

then, ∂ restricts to a derivation on the category of *J*-sheaves (that is itself monoidal).

Derivatives of models

Let $\mathcal{T}$ be an algebraic theory of some sort, with the property that *Mod*$(\mathcal{T})$ is monoidal; let $\partial : \hat{\mathcal{T}} \to \hat{\mathcal{T}}$ be a convolution-derivation; if ∂ is such that

$$\boxed{\text{a big black hole of ignorance}}$$

then ∂ sends a $\mathcal{T}$-model to a $\mathcal{T}$-model.