Coends of higher arity

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Overview

Recently appeared on arXiv @

- j/w with Théo de Oliveira Santos
- stemmed from previous Itaca (thanks Alessio!)
- (but slight variation on his work, tweaks here and there)
- still an ongoing project (likely two spinoffs, " Diagonal category theory " and " Weighted coends ")
- examples take *a lot* of space. Like, a lot. Let's discuss them later.

Tensor calculus

In 1916 a modest employee at Bern's patent office has an interesting idea: many computations in differential geometry become simpler if we suppress the summation symbol, implicitly understood between a 'covariant' and a 'contravariant' index: terrifying expressions like

$$
\begin{aligned}
R_{ij} = &-\sum_{a,b} \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a}\right) g^{ab} \\
&+ \frac{1}{2} \sum_{a,b,c,d} \left(\frac{1}{2}\frac{\partial g_{ac}}{\partial x^i}\frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jb}}{\partial x^d}\right) g^{ab} g^{cd} \\
&- \frac{1}{4} \sum_{a,b,c,d} \left(\frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c}\right)\left(2\frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\right) g^{ab} g^{cd}.
\end{aligned}
$$

become more compact.

Coend calculus

In category theory, "summing over repeated indices" corresponds to taking a coend of a functor

$$T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

intended as the quotient of $\coprod_C T(C, C)$ by the equivalence relation generated by the action of T on arrows;

if $S : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Set}$ and $T : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ are two profunctors, their composition

$$(S \diamond T)(A, C) := \int^B S(A, B) \times T(B, C)$$

is akin to the matrix product of two matrices.

Questions

- What if we want to sum/integrate/coend over an "unbalanced tensor" like
$$T : (\mathcal{C}^{\text{op}})^p \times \mathcal{C}^q = \mathcal{C}^{(p,q)} \to \mathcal{D}$$
for $p, q \geq 1$?
Yes, there are coends for "higher arity" functors $\mathcal{C}^{(p,q)} \to \mathcal{D}$;
- Is the resulting theory well-behaved as the classical one?
Yes and no; HA co/ends are particular instances of co/ends, where the integrand T has been "completely symmetrised";
- Is the resulting theory useful as the classical one?
The resulting theory is kinda expressive, and captures some new phenomena.

Example (Street-Dubuc)

Let $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be two functors; define

$$\mathrm{DNat}(F^{\uparrow}, G^{\downarrow}) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$

sending (A, B) to $\mathcal{D}(F(B, A), G(A, B))$; then, the set of dinatural transformations $F \rightsquigarrow G$ is canonically isomorphic to the end of $\mathrm{DNat}(F^{\uparrow}, G^{\downarrow})$, i.e. to the equaliser of the diagram

$$\prod_C \mathcal{D}(F(C, C), G(C, C)) \underset{v}{\overset{u}{\rightrightarrows}} \prod_{A \to B} \mathcal{D}(F(B, A), G(A, B))$$

Proposition

$$\mathrm{DNat}(F, G) \cong {}_{(2,2)}\!\int_A \mathrm{DNat}(F(A, A), G(A, A))$$

A notion of generalised dinaturality for $\varphi_{\underline{A},\underline{A}} : F(\underline{A},\underline{A}) \rightsquigarrow G(\underline{A},\underline{A})$ recently introduced by A. Santamaria.

Slightly too general for our purposes for two reasons

- naturality as a property of a single component (φ can be di/natural at an index i, but “unnatural” elsewhere).
- There are ‘dinatural transformations’ between functors of different arity, $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$, $G : \mathcal{C}^{(r,s)} \to \mathcal{D}$.

Main ingredient

Functors $\mathcal{C}^{\alpha} \to \mathcal{B}$, where α is a "binary multi-index", i.e. a string of $\{\oplus, \ominus\}$'s, and we put $\mathcal{C}^{\varnothing} :=$, the terminal category, $\mathcal{C}^{\oplus} := \mathcal{C}$, $\mathcal{C}^{\ominus} := \mathcal{C}^{\mathrm{op}}$, and $\mathcal{C}^{\alpha \uplus \alpha'} := \mathcal{C}^{\alpha} \times \mathcal{C}^{\alpha'}$.

Convention

A generic power $\mathcal{C}^{\alpha}$ is always "reshuffled" in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right.

$$F : \mathcal{C}^{(p,q)} \to \mathcal{D} \quad \text{is a functor of "type"} \begin{bmatrix} p \\ q \end{bmatrix}$$

Notation

- A generic tuple of objects,

$$\underline{A} := (A_1, \ldots, A_n)$$

often split as the juxtaposition $\underline{A}'; \underline{A}''$ of two subtuples of length p, q,

$$\underline{A}' := (A_1, \ldots, A_q), \qquad \underline{A}'' := (A_{p+1}, \ldots, A_{p+q})$$

- Contravariant components are always *left* in the typing

$$F : \mathcal{C}^{(p,q)} \to \mathcal{D}$$

of a functor, and *up* in its action on objects.

- Evaluate a functor F of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ at a tuple of identical objects:

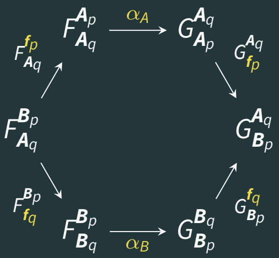
$$F^{\boldsymbol{A}}_{\boldsymbol{A}} := F^{A,\ldots,A}_{A,\ldots,A},$$

Definition

Let F, G be of type $\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} q \\ p \end{bmatrix}$. Then a (p,q)*-to-*(q,p)*-dinatural transformation* $\alpha : F \rightsquigarrow G$ is a collection

$$\{\alpha_A : F^{\overbrace{A,\dots,A}^{p \text{ times}}}_{\underbrace{A,\dots,A}_{q \text{ times}}} \longrightarrow G^{\overbrace{A,\dots,A}^{q \text{ times}}}_{\underbrace{A,\dots,A}_{p \text{ times}}} \mid A \in \mathcal{C}_o\}$$

of morphisms of $\mathcal{D}$ indexed by the objects of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, this diagram commutes:

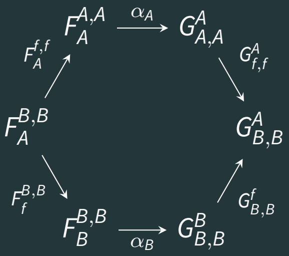


Example

For (p, q)=$(2, 1)$, a $(2, 1)$-dinatural transformation is a collection

$$\left\{ \alpha_A : F_A^{A,A} \to G_{A,A}^A \,\middle|\, A \in \mathcal{C}_o \right\}$$

of morphisms of $\mathcal{D}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the hexagon commutes:



What's with the (p, q)-to-(q, p)?

Santamaria's definition is "(p, q)-to-(r, s)" dinaturality; we could have stick to the stricter notion of (p, q)-to-(p, q) dinaturality.

But then some theorems would have been tricky to state.

Our definition sits in the middle:

> *the type of domain and codomain of a "higher arity" dinatural transformation $\alpha : F \rightsquigarrow G$ are different, but just swapped: the contravariant length of F is the covariant length of G, and vice-versa.*

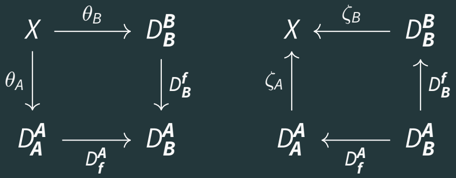
(wedges and cowedges are the same, tho)

No surprise

Definition

Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor and let $X \in \mathcal{D}_o$.

- A (p, q)-wedge for D under X is a (p, q)-dinatural transformation $\theta : X \rightsquigarrow D$ from the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X to D;
- A (p, q)-cowedge for D over X is a (p, q)-dinatural transformation $\zeta : D \rightsquigarrow X$ from D to the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X.



No surprise, II

- The (p,q)-end of $T : \mathcal{C}^{(p,q)} \to \mathcal{D}$ if it exists, is the terminal object in the category of (p,q)-wedges;
- The (p,q)-coend of $T : \mathcal{C}^{(p,q)} \to \mathcal{D}$ if it exists, is the initial object in the category of (p,q)-cowedges;

Basic properties of (p, q)-co/ends

- Functoriality Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor. Sending D to its co/end is a functor

$$
{}_{(p,q)}\int_{A\in\mathcal{C}} : \mathrm{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right) \longrightarrow \mathcal{D},
$$

$$
{}^{(p,q)}\int^{A\in\mathcal{C}} : \mathrm{Cat}\left(\mathcal{C}^{(p,q)}, \mathcal{D}\right) \longrightarrow \mathcal{D}
$$

Basic properties of (p, q)-co/ends

- (p, q)-Wedges and (p, q)-diagonals For each $X \in \mathcal{C}_o$ we have natural bijections

$$\mathrm{Wd}^{(p,q)}_{(-)}(D) \cong \mathrm{Wd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big),$$
$$\mathrm{CWd}^{(p,q)}_{(-)}(D) \cong \mathrm{CWd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big).$$

where $\Delta_{p,q}$ is the "twisted diagonal" functor

$$\Delta_{p,q} := \underbrace{\Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}}_{p \text{ times}} \times \underbrace{\Delta \times \cdots \times \Delta}_{q \text{ times}}.$$

and $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ = diagonal.

Basic properties of (p, q)-co/ends

- (p, q)-Ends as ordinary ends We have natural isomorphisms

$$
{}^{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} \cong \int_{A\in\mathcal{C}} \Delta_*^{(p,q)}(D)^A_A,
$$
$$
{}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{A} \cong \int^{A\in\mathcal{C}} \Delta_*^{(p,q)}(D)^A_A.
$$

where $\Delta_{p,q}$ is the twisted diagonal functor. The (p, q)-end functor factors as

$$
\mathsf{Fun}(\mathcal{C}^{(p,q)}, \mathcal{D}) \xrightarrow{\Delta_*^{(p,q)}} \mathsf{Fun}(\mathcal{C}^{\mathsf{op}} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\int_A} \mathcal{D},
$$

and similarly so do (p, q)-coends.

Basic properties of (p,q)-co/ends

- (p,q)-Ends as limits There are co/equaliser diagrams

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \longrightarrow \prod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \overset{\lambda}{\underset{\rho}{\rightrightarrows}} \prod_{A\to B} D^{\underline{A}}_{\underline{B}} $$

$$ \coprod_{A\to B} D^{\underline{A}}_{\underline{B}} \overset{\lambda'}{\underset{\rho'}{\rightrightarrows}} \coprod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \longrightarrow {}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} $$

for suitable maps $\lambda, \rho, \lambda', \rho'$, induced by the morphisms $D^{\underline{A}}_{\underline{u}}, D^{\underline{u}}_{\underline{B}}$.

Basic properties of (p, q)-co/ends

- (p, q)-Ends as limits, yet again There exists a category $\mathrm{Tw}^{(p,q)}(\mathcal{C})$ together with a universal fibration

$$\Sigma\colon \mathrm{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \mathcal{C}^{(p,q)}$$

inducing natural isomorphisms

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} \cong \lim\left(\mathrm{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\right),$$

$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}} \cong \operatorname{colim}\left(\mathrm{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\right).$$

Basic properties of (p,q)-co/ends

- Commutativity of (p,q)-ends with homs We have isomorphisms

$$\mathcal{D}\left(X, {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}\right) \cong {}_{(p,q)}\int_{A\in\mathcal{C}} \mathcal{D}\left(X, D^{A}_{\underline{A}}\right)$$

$$\mathcal{D}\left(\,{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{A}_{\underline{A}}, X\right) \cong {}_{(q,p)}\int_{A\in\mathcal{C}} \mathcal{D}\left(D^{A}_{\underline{A}}, X\right).$$

natural in $X \in \mathcal{D}$.

Theorem (Fubini, 1907)
Let $D : \mathcal{A}^{(p,q)} \times \mathcal{B}^{(r,s)} \longrightarrow \mathcal{D}$ *be a functor. Then*

$$ {}_{(p+r,q+s)}\int_{(A,B)} D^{\mathbf{A,B}}_{\mathbf{A,B}} \cong {}_{(p,q)}\int_A {}_{(r,s)}\int_B D^{\mathbf{A,B}}_{\mathbf{A,B}} \cong {}_{(r,s)}\int_B {}_{(p,q)}\int_A D^{\mathbf{A,B}}_{\mathbf{A,B}}, $$

$$ {}^{(p+r,q+s)}\int^{(A,B)} D^{\mathbf{A,B}}_{\mathbf{A,B}} \cong {}^{(p,q)}\int^A {}^{(r,s)}\int^B D^{\mathbf{A,B}}_{\mathbf{A,B}} \cong {}^{(r,s)}\int^B {}^{(p,q)}\int^A D^{\mathbf{A,B}}_{\mathbf{A,B}} $$

as objects of $\mathcal{D}$.

Proof.
Coending is a left adjoint; left adjoints compose. Dually for ends. □

Fubini does not reduce arity

Note that p, q, r, s can't be broken in the sum of smaller integers:

That is, the Fubini rule does not allow us to reduce the arity of a higher arity co/end when $\mathcal{A} = \mathcal{B}$:

$$_{(p,q)}\int_{A} {}_{(r,s)}\int_{B} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}_{(p+r,q+s)}\int_{(A,B)\in\mathcal{A}\times\mathcal{A}} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \ncong {}_{(p+r,q+s)}\int_{A\in\mathcal{C}} D^{\boldsymbol{A}}_{\boldsymbol{A}}.$$

the point being that we are integrating over a pair (A, B), and not over a single variable A.

later

Examples are for ~~weaklings~~

Some (p, q)-co/ends are trivial for trivial reasons

- Let R be a ring;

$$ {}^{(0,2)}\!\!\int_{A\in Mod_R} A\otimes A \cong 0, \qquad {}^{(0,2)}\!\!\int^{A\in Mod_R} A\otimes A \cong 0 $$

- Fact: If $\mathcal{C}$ is a sifted category, all diagonal functors $\Delta : \mathcal{C} \to \mathcal{C}^n$ are final, because the product and composition of final functors is itself final.

- Fact: Let $X : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta} \to \mathrm{Set}$ be a bisimplicial set; then

$$ {}^{(2,0)}\!\!\int^{[n]\in\mathbf{\Delta}} X_{n,n} \cong \pi_0(\mathrm{d}(X)), \qquad {}_{(2,0)}\!\!\int_{[n]\in\mathbf{\Delta}} X_{n,n} \cong X_{0,0}. $$

- ((p, q)-Street-Dubuc) Let $F, G : \mathcal{C}^{(p,q)} \to \mathcal{D}$; then

$$ \mathrm{DiNat}^{(p,q)}(F, G) \cong {}_{(p,q)}\!\!\int_C \mathcal{D}(F^C_C, G^C_C) $$

Kusarigamas

Kusarigama

Kusarigamas are correspondences

$$\daleth^{p,q} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathsf{Cat}\big(\mathcal{C}^{(q,p)}, \mathcal{D}\big),$$

$$\Gamma^{p,q} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathsf{Cat}\big(\mathcal{C}^{(q,p)}, \mathcal{D}\big),$$

that can be regarded as a universal way to 'switch' the type of a functor; more importantly, they connect naturality with dinaturality:

$$\mathsf{DiNat}^{(p,q)}(F, G) \cong \mathsf{Nat}\left(F, \Gamma^{p,q}(G)\right) \cong \mathsf{Nat}\left(\daleth^{p,q}(F), G\right);$$

(thus $\daleth^{p,q} \dashv \Gamma^{p,q}$)

Kusarigama

The paramount property of the co/kusarigama functors is that

> *given $\mathcal{C}$, the category of elements of $\mathbb{I}^{p,q}(1)$, where $1 : \mathcal{C}^{(p,q)} \to \mathbf{Set}$ chooses 1, is the universal fibration building a higher-arity version of twisted arrow categories.*

This makes it possible to express the (p,q)-co/end of $G : \mathcal{C}^{(p,q)} \to \mathcal{D}$ as a co/limit over the (p,q)-twisted arrow category of $\mathcal{C}$:

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \lim\Big(\mathrm{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), $$

$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \operatorname{colim}\Big(\mathrm{Tw}^{(p,q)}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big). $$

...Why "kusarigama"?

Suppose that $\mathcal{D}$ is cocomplete (resp., complete); then,

$$\text{⅃}^{p,q}(F)(\underline{X},\underline{Y}) \cong {}^{(q,p)}\!\!\int^{A\in\mathcal{C}} \left(\mathsf{h}^{A}_{X_1}\times\cdots\times\mathsf{h}^{A}_{X_p}\times\mathsf{h}^{Y_1}_{A}\times\cdots\times\mathsf{h}^{Y_q}_{A}\right)\odot F^{A,\dots,A}_{A,\dots,A}$$

$$\Gamma^{p,q}(G)(\underline{X},\underline{Y}) \cong {}_{(q,p)}\!\!\int_{A\in\mathcal{C}} \left(\mathsf{h}^{A}_{X_1}\times\cdots\times\mathsf{h}^{A}_{X_p}\times\mathsf{h}^{Y_1}_{A}\times\cdots\times\mathsf{h}^{Y_q}_{A}\right)\pitchfork G^{A,\dots,A}_{A,\dots,A}$$

$$\Gamma^{q,p}(G) := \int_{A\in\mathcal{C}} \mathsf{h}_A\times\cdots\times\mathsf{h}_A\times\mathsf{h}^A\times\cdots\times\mathsf{h}^A\pitchfork G^A_A.$$

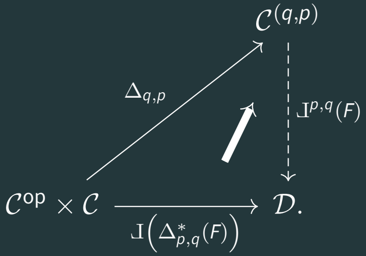
Inductive definition

The cokusarigama

$$\daleth^{p,q}(F) \colon \mathcal{C}^{(q,p)} \longrightarrow \mathcal{D}$$

of a functor $F \colon \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ is the left Kan extension of the $(1,1)$-cokusarigama of $\Delta^*_{p,q}(F)$ along $\Delta_{q,p}$:

$$\daleth^{p,q}(F) = Lan_{\Delta_{q,p}} \left(\daleth\left(\Delta^*_{p,q}(F)\right)\right)$$



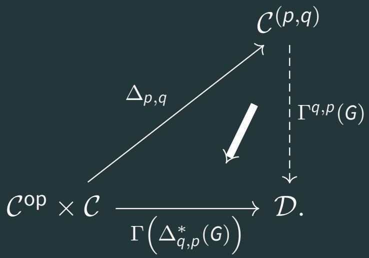
Inductive definition

Dually, the kusarigama

$$\Gamma^{q,p}(G)\colon \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$$

of $G\colon \mathcal{C}^{(q,p)} \longrightarrow \mathcal{D}$ is the right Kan extension of the $(1,1)$-kusarigama of $\Delta^*_{q,p}(G)$ along $\Delta_{p,q}$:

$$\Gamma^{q,p}(G) = Ran_{\Delta_{p,q}}\left(\Gamma\left(\Delta^*_{q,p}(G)\right)\right)$$



(p,q)-twisted category

Definition

The (p,q)-*twisted arrow category* is the category $\mathrm{Tw}^{(p,q)}(\mathcal{C})$ defined as the category of elements $\mathcal{C}^{(q,p)} \int ⅃^{p,q}(1)$ of $⅃^{p,q}(1)$:

$$\begin{array}{ccc}
\mathrm{Tw}^{(p,q)}(\mathcal{C}) & \xrightarrow{\Sigma_{(p,q)}} & \mathcal{C}^{(p,q)} \\
\downarrow & \nearrow & \downarrow ⅃^{p,q}(1) \\
1 & \xrightarrow[1]{} & \mathsf{Set}.
\end{array}$$

(p, q)-twisted category

If $\mathcal{C}$ has finite products and coproducts, we gain an additional equivalent description of $\mathrm{Tw}^{(p,q)}(\mathcal{C})$:

1. The category whose
    - Objects are morphisms $A_1 \coprod \cdots \coprod A_p \longrightarrow B_1 \times \cdots \times B_q$;
    - Morphisms are factorisations of the codomain through the domain, of the form

$$\begin{array}{ccc} A_1 \coprod \cdots \coprod A_p & \xrightarrow{f} & B_1 \times \cdots \times B_q \\ {\scriptstyle \phi_1 \coprod \cdots \coprod \phi_p} \uparrow & & \downarrow {\scriptstyle \psi_1 \times \cdots \times \psi_q} \\ A'_1 \coprod \cdots \coprod A'_p & \xrightarrow[g]{} & B'_1 \times \cdots \times B'_q. \end{array}$$

Future prospects

Where is this going?

Weighted coends stand to co/ends in the same relation as weighted co/limits stand to limits.

Definition (Weighted co/end)

Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-enriched categories and $D\colon \mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{V}} \mathcal{C} \longrightarrow \mathcal{D}$ a $\mathcal{V}$-functor, and $W : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ a $\mathcal{V}$-presheaf.

1. The *end of D weighted by W* is such that

$$\hom_{\mathcal{D}}\left(X, \int_{A\in\mathcal{C}}^{W} D_A^A\right) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_{\mathcal{C}}(X, D))$$

2. The *coend of D weighted by W* is such that

$$\hom_{\mathcal{D}}\left(\int_{W}^{A\in\mathcal{C}} D_A^A, Y\right) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_{\mathcal{C}}(D, Y))$$

Where is this going?

Turns out that:

- There are examples of this construction;
- the weighted end of $D\colon \mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{V}} \mathcal{C} \longrightarrow \mathcal{D}$ by $W : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ is the (2,2)-end
$$ {}_{(2,2)}\int_C W(C,C) \pitchfork T(C,C) $$
- One can define a whole new world of categorical concepts, weighing Kan extensions, natural transformations, monads…

Where is this going?

Diagonal category theory Replace ‘natural’ with ‘dinatural’ in most categorical concepts:

cons: dinaturals do not compose.

pros: you shouldn’t care: one can exhibit ‘weighted composition maps’

$$\mathrm{DiNat}(G, H) \times \mathrm{DiNat}(F, G) \longrightarrow \mathrm{Nat}^{[\mathrm{h},\mathrm{h}]}(F, H),$$

where the latter set is to dinatural transformations as dinatural transformations are to natural transformations! More generally, we have a diagonality hierarchy,

$$\mathrm{Nat}(F, G), \mathrm{DiNat}(F, G), \mathrm{Nat}^{\mathrm{h},\mathrm{h}}(F, G), \mathrm{Nat}^{\mathrm{h},\mathrm{h},\mathrm{h}}(F, G), \mathrm{Nat}^{\mathrm{h},\mathrm{h},\mathrm{h},\mathrm{h}}(F, G), \ldots$$

together with compositions

$$\mathrm{Nat}^{\mathrm{h}^n}(G, H) \times \mathrm{Nat}^{\mathrm{h}^m}(F, G) \longrightarrow \mathrm{Nat}^{\mathrm{h}^{n+m}}(F, H).$$

Free space for discussion

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