Differential 2-rigs

2-(commutative algebra)

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- Started in 2021 with a simple question: what is a monoidal category equipped with a functor that is linear and Leibniz?

$$\partial(A+B)\cong\partial A+\partial B \qquad \partial(A\otimes B)\cong\partial A\otimes B+A\otimes\partial B$$

- j/w Todd Trimble who wanted to understand a mysterious paper

CYCLIC OPERADS AND CYCLIC HOMOLOGY

E. GETZLER AND M.M. KAPRANOV

The cyclic homology of associative algebras was introduced by Connes [4] and Tsygan [22] in order to extend the classical theory of the Chern character to the non-commutative setting. Recently, there has been increased interest in more general algebraic structures than associative algebras, characterized by the presence of several algebraic operations. Such structures appear, for example, in homotopy theory [18], [3] and topological field theory [9]. In this paper, we extend the formalism of cyclic homology to this more general framework.

- so far, **: 10.4204/EPTCS.380.10**, **: 2401.04242**

Notions of 2-rig

A ring is

$$(R, +), (R, \cdot)$$
$$a \cdot (x + y) = a \cdot x + a \cdot y \qquad (x + y) \cdot a = \ldots$$
$$a \cdot 1 = 1 \cdot a = a$$

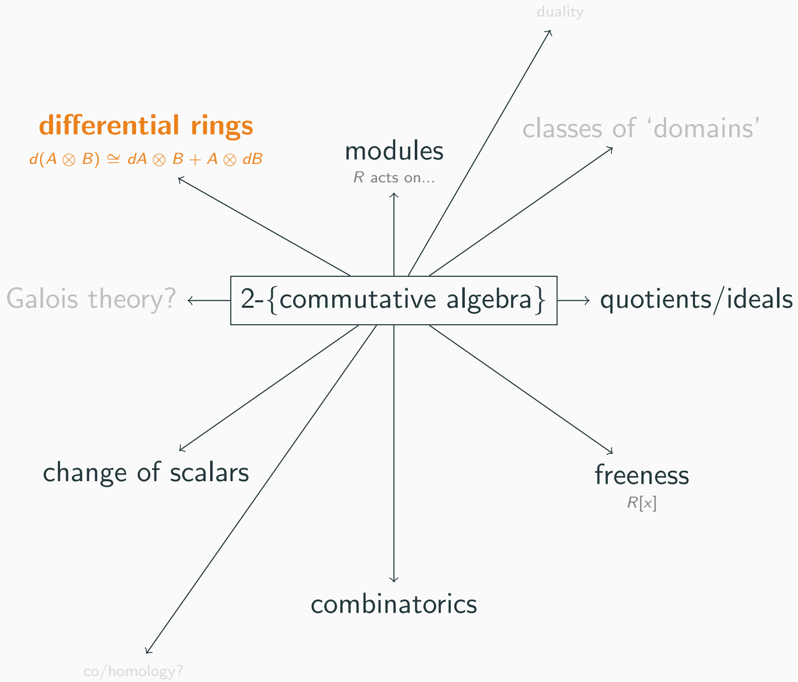
Competing definition

- [Laplaza] a monoidal category with two tensors $\oplus, \otimes$, distributing one over the other
  - ☺ language good (lots of examples)
  - ☹ coherence bad

    23 axioms, Elgueta paper...
- [Baez-Moeller-Trimble] a symmetric monoidal, k-enriched, Cauchy complete category
  - nice applications to species and Schur functors (that's why Todd got in touch with me!)
- [**L**] a (symmetric) monoidal category $(\mathcal{R}, \otimes)$ where each $A \otimes -$ commutes with coproducts (or with a prescribed class of colimits)
  - (compare w/ Laplaza)

Some examples:

- Distributive categories (where $\otimes = \times$)
- presheaves over $(\mathcal{C}, \otimes)$, with Day convolution
- vector bundles over a manifold
- κ-presentable tensor categories, where κ-presentable objects are closed under κ-colimits
- ...


duality
differential rings
$d(A \otimes B) \cong dA \otimes B + A \otimes dB$
modules
R acts on...
classes of ‘domains’
Galois theory?
2-{commutative algebra}
quotients/ideals
change of scalars
freeness
$R[x]$
combinatorics
co/homology?

The case of species

Species

Let S be a set. Let $\mathcal{V}$ be a symmetric monoidal closed, complete and cocomplete category (a 'cosmos').

Regard S as a discrete category, let $\mathbf{P}[S]$ be the free symmetric monoidal category on S.

Definition

The category $(S, \mathcal{V})$-$\mathbf{Spc}$ of $(S, \mathcal{V})$-species is the category of functors $\mathbf{P}[S] \to \mathcal{V}$ and natural transformations.

For today, $S = \{*\}$ is a singleton, and $\mathcal{V} = \mathbf{Set}$. Other choices are possible (e.g., $\mathcal{V} = \mathbf{Vect}_k$ is probably the version algebraic topologists are more familiar with). Then

$$\boxed{\mathbf{P} := \mathbf{P}[1] \qquad \mathbf{Spc} := (1, \mathbf{Set})\text{-}\mathbf{Spc} = [\mathbf{P}, \mathbf{Set}]}$$

Species

- **Spc** is the category of copresheaves on **P**, the groupoid of natural numbers: objects finite sets $[n]$, morphisms bijections (in partic. $\mathbf{P}([n],[m]) = \varnothing$ if $n \neq m$)
- Rich supply of monoidal structure(s) interacting with each other (esp. when instead of **Set**-presheaves one takes k-linear presheaves)
- **Spc** is equipped with $\partial : \mathbf{Spc} \to \mathbf{Spc}$ that 'shifts' a functor by 1, $F'[n] := F[n+1]$
- Leibniz rule $(F \otimes G)' \cong F' \otimes G + F \otimes G'$ (Day convolution)
- Chain rule $(F \circ G)' \cong (F' \circ G) \otimes G'$ (operadic or 'plethystic' composition)
- $L \dashv \partial \dashv R$ (this is important and nontrivial!)

Let me expand...

A species is a functor $F : \mathbf{P} \to \mathbf{Set}$, or equiv. a family of right S_n-sets X_n:

$$\mathbf{Cat}(\mathbf{P}, \mathbf{Set}) \cong \mathbf{Cat}\big(\sum_{n\geq 0} S_n, \mathbf{Set}\big) \cong \prod_{n\geq 0} \mathbf{Cat}(S_n, \mathbf{Set})$$

Examples of species:

- The species E of elements; constant at the singleton / S_n action is always trivial
- The species P of subsets; sends $[n]$ to $2^n = \{U \subseteq [n]\}$ / S_n action is by permuting a subset
- The species Sym of permutations; sends $[n]$ to S_n / S_n action is by multiplication
- The species L of linear orders; sends $[n]$ to the set of linear orders L_n on $[n]$ / S_n action is by conjugation
- The species Cyc of cyclic orders, def'd similarly.

- $[n] \oplus [m] := [n+m]$ defines a (symmetric) monoidal structure on $\mathbf{P}$;
- **Spc** inherits a Day convolution (symmetric, closed) monoidal structure

$$\mathbf{Spc}(F * G, H) \cong \mathbf{Spc}(F, \{G, H\})$$

- There is a functor $\partial : \mathbf{Spc} \to \mathbf{Spc}$ defined by 'shifting F by 1'

Try to prove the Leibniz rule!

$$f(X) = \sum_{n \geq 0} \frac{a_n}{n!} X^n \qquad F[X] = \sum_{n \geq 0} \frac{F[n]}{\sim_{S_n}} X^n$$

$$\frac{d}{dX} f(X) = \sum_{n \geq 0} \frac{a_{n+1}}{(n+1)!} X^n \qquad \partial F[X] = \sum_{n \geq 0} \frac{F[n+1]}{\sim_{S_{n+1}}} X^n$$

- $E' \cong E \qquad Cyc' \cong L \qquad P' \cong E + E$
- ∂ has a left adjoint (easy to describe: $\partial = \{y[1], -\}$ hence $L = y[1] * -$), but also a right adjoint (because $y[1]$ is a tiny object)

D2Rs

This motivates the definition of a differential 2-rig (D2R):

A 2-rig $(\mathcal{R}, \otimes)$ equipped with an endofunctor $\partial : \mathcal{R} \to \mathcal{R}$ such that

- $\partial(A + B) \cong \partial A + \partial B$
- $\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$

Equivalently: ∂ is equipped with two tensorial strengths, forming a coproduct diagram

$$\partial A \otimes B \to \partial(A \otimes B) \leftarrow A \otimes \partial B$$

This realizes the Leibniz rule as a universal property.

D2Rs

The formulation with tensorial strength is very useful for bookkeeping coherences of ∂:

LR1) naturality: the diagram

$$\begin{array}{ccc}
\partial(A \otimes B) & \xrightarrow{\partial(u \otimes v)} & \partial(A' \otimes B') \\
\uparrow{\scriptstyle \iota_{AB}} & & \uparrow{\scriptstyle \iota_{A'B'}} \\
\partial A \otimes B + A \otimes \partial B & \xrightarrow[\partial u \otimes v + u \otimes \partial v]{} & \partial A' \otimes B' + A' \otimes \partial B'
\end{array} \tag{1.10}$$

is commutative for every pair of morphisms $u : A \to A'$ and $v : B \to B'$ in $\mathcal{C}$.

LR2) compatibility with the right distributor:

$$\begin{array}{ccc}
\partial((Y+Z) \otimes X) & \xleftarrow{\partial \delta^{R}} & \partial(Y \otimes X + Z \otimes X) \\
\uparrow{\scriptstyle \iota} & & \downarrow \\
(Y+Z) \otimes \partial X + \partial(Y+Z) \otimes X & & \partial(Y \otimes X) + \partial(Z \otimes X) \\
\uparrow{\scriptstyle \delta^{R} + \delta^{R}} & & \uparrow{\scriptstyle \iota + \iota} \\
Y \otimes \partial X + Z \otimes \partial X + \partial Y \otimes X + \partial Z \otimes X & \longrightarrow & \partial Z \otimes X + Z \otimes \partial X + \partial Z \otimes Y + Z \otimes \partial Y
\end{array}$$

LR3) compatibility with the left distributor:

Freeness results

Spc is the free (cocomplete) 2-rig $F[t]$ on a single generator $\{t\}$; it acquires a differential structure much like $k[x]$ does.

Spc is also initial among cocomplete 2-rigs.

The free differential 2-rig on a single generator is also a category of species:

$$F_\partial[Y, Y', Y'', \dots] \cong \mathbf{Set}^{\mathbf{P}[y_0, y_1, y_2, \dots]}$$

Free 2-rig on a category...

Free $\mathcal{R}$-algebra on S: $F[S] \otimes \mathcal{R}$

$\mathcal{R}$ a 2-rig; $\mathcal{R}[t] = \mathcal{R} \otimes_{\mathbf{P}} F[t] = \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc}$

Geometry of D2Rs

$$\{\text{derivations on } R\} \cong \left\{ s : \begin{array}{c} R[t]/t^2 \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle ev_0} \\ R \end{array} \right\}$$

Kähler differentials

$$\{\text{derivations on } \mathcal{R}\} \cong \hom_{\textbf{2-Rig}}(\mathcal{R}, \mathcal{R}[t]/(t^2))$$

$$\mathcal{R}[t]/(t^2) \cong \text{coinverter}\Big(\mathcal{R}[t] \underset{-\otimes t^2}{\overset{\varnothing}{\rightrightarrows}} \mathcal{R}[t] \Big) \quad (\Downarrow)$$

a certain kind of 2-dimensional colimit

Intermezzo

Definition

Given a 2-category $\mathcal{K}$ and a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \quad (\alpha : f \Rightarrow g)$$

the coinverter of f, g is a 1-cell $c : B \to Q$ such that

- $c * \alpha : cf \Rightarrow cg$ is invertible;
- (Q, c) is 1-initial and 2-initial among such pairs.

Let $\mathcal{C}$ be a category, $W \subseteq \mathcal{C}^2$ a class of maps; the coinverter of

$$(W \subseteq \mathcal{C}^2) \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C} \quad (\alpha : s \Rightarrow t)$$

is the homotopy category $\mathcal{C}[W^{-1}]$.

Geometry of D2Rs

Let $\mathcal{R}$ be a cocomplete 2-rig.

Consider the unique 2-cell $\varnothing \Rightarrow (- \otimes t^2)$, where $- \otimes t^2$ 'multiplies by t^2'; the coinverter

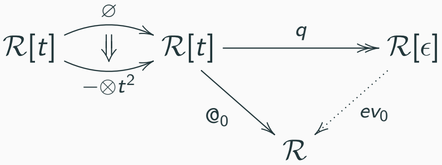
$$\mathcal{R}[t] \underset{-\otimes t^2}{\overset{\varnothing}{\underset{\Downarrow}{\rightrightarrows}}} \mathcal{R}[t] \xrightarrow{q} C$$

coincides with the procedure of killing off polynomials divisible by t^2, hence C is the 'quotient 2-rig' by the ideal (t^2).

$$(a + tb)(c + td) = ac + (ad + bc)t + \cancel{t^2 bd}$$

Geometry of D2Rs

Now we would like to build the 'space of sections' of a canonical 'evaluation at 0' map



$@_0 : \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc} \to \mathcal{R} : (A, F) \mapsto \sum_{F[0]} A$ is induced by the universal property of coproducts.

Theorem

$$Der[\mathcal{R}] \cong \{\text{sections}/\mathcal{R} \text{ of } ev_0 : \mathcal{R}[\epsilon] \to \mathcal{R}\}$$

Geometry of D2Rs

- similarly: quotient for a principal ideal, say $\mathfrak{I} = (p)$, is coinverter of

$$\mathcal{R}[t] \underset{-\otimes p(t)}{\overset{\varnothing}{\underset{\Downarrow}{\rightrightarrows}}} \mathcal{R}[t] \overset{q}{\twoheadrightarrow} \mathcal{R}[t]/(p)$$

- Ideals are easy to define, but
  - Domains? $A \otimes B \cong \varnothing \Rightarrow A = B = \varnothing$?
  - quotient for a non-principal ideal $\mathfrak{I} = (p_i \mid i \in I)$ is a...?
  - What's a 2-PID?
- quotients like $\mathcal{R}[X, Y]/(Y^2 + 1 \cong X^2)$ (categorified hyperbola) acquire a differential structure, $\partial Y = X, \partial X = Y$; can be done more in general?

Jet spaces

Categorified jet spaces

Given a D2R $(\mathcal{R}, \otimes, \partial)$ let $\mathbf{Alg}(\partial)$ be the category of ∂-algebras.

- objects: $(X, \xi : \partial X \to X)$;
- morphisms: $f : X \to Y$ compatible with the structure map.

$\mathbf{Alg}(\partial)$ is itself a 2-rig and ∂ lifts to a derivation ∂' on $\mathbf{Alg}(\partial)$. Hence the chain

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{Alg}(\partial') & \longrightarrow & \mathbf{Alg}(\partial) & \longrightarrow & \mathcal{R} \\ & & \downarrow{\scriptstyle \partial''} & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial} \\ \dots & \longrightarrow & \mathbf{Alg}(\partial') & \longrightarrow & \mathbf{Alg}(\partial) & \longrightarrow & \mathcal{R} \end{array}$$

Categorified jet spaces

Define by mutual induction:

- $\mathcal{R}^{(0)} := \mathcal{R}$ and $\mathcal{R}^{(n+1)} := \mathbf{Alg}(\partial^{(n)}, \mathcal{R}^{(n)})$;
- $\partial^{(1)} := \partial$ and $\partial^{(n+1)} := \mathcal{R}^{(n+1)} \to \mathcal{R}^{(n+1)}$ defined lifting $\partial^{(n)}$.

Chain of forgetful functors

$$\mathcal{R} \longleftarrow \mathbf{Alg}(\partial) \longleftarrow \mathbf{Alg}(\partial') \longleftarrow \mathbf{Alg}(\partial'') \longleftarrow \cdots$$

$$\mathbf{Jet}[\mathcal{R}, \partial] := \lim \left(\mathcal{R} \xleftarrow{U} \mathcal{R}^{(1)} \xleftarrow{U^{(1)}} \mathcal{R}^{(2)} \xleftarrow{U^{(2)}} \cdots \right).$$

Categorified jet spaces

The typical object in $\mathbf{Jet}[\mathcal{R}, \partial]$ consists of a countable sequence

$$\vec{X} = \big(X, (X; \xi : \partial X \to X), ((X;\xi); \xi' : \partial'(X;\xi) \to (X;\xi)), \dots\big)$$

the n^{th} element of which equips the $(n-1)^{\text{th}}$ with an algebra structure for $\partial^{(n)}$.

$$X \xleftarrow{\xi} \partial X \xleftarrow{\xi'} \partial\partial X \xleftarrow{\xi''} \partial\partial\partial X \leftarrow \dots$$

Categorified jet spaces

Define the *k-jet* $J^k(\vec{X})$ of an object $\vec{X} \in \mathbf{Jet}[\mathcal{R}, \partial]$ as the image of $\vec{X}$ under the functor J^k obtained from the limit projections $\pi_k : \mathbf{Jet}[\mathcal{R}, \partial] \to \mathcal{R}^{(k)}$ as

$$J^k := \langle \pi_0, \ldots, \pi_k \rangle : \mathbf{Jet}[\mathcal{R}, \partial] \longrightarrow \prod_{i=0}^{k} \mathcal{R}^{(i)}$$

cf. differential geometry, where the k-jet of a real valued function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$(J^k_{x_0} f)(z) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(x_0)}{\ell!} z^\ell = f(x_0) + f'(x_0)z + \cdots + \frac{f^{(k)}(x_0)}{k!} z^k$$

Differential operators

Differential operators

- Let $\mathcal{R}$ be a 2-rig; denote **Der**$[\mathcal{R}]$ the category of derivations of $\mathcal{R}$;
- as such, it's a full subcategory of the 2-rig $\mathbf{Cat}_+(\mathcal{R}, \mathcal{R})$
- and a left $\mathcal{R}$-module, $\lambda X.A \otimes \partial X \in \mathbf{Der}[\mathcal{R}]$.

It is in general quite difficult to determine the structure of **Der**$[\mathcal{R}]$; something can be said for species.

Differential operators

Assume $(\mathcal{R}, \otimes)$ is monoidal closed, and differential wrt $\partial : \mathcal{R} \to \mathcal{R}$; let $L \dashv \partial$; then:

Theorem

Consider the following conditions:

1. $\{LX, Y\} \cong \{X, \partial Y\}$ naturally;
2. $L(X \otimes Y) \cong LX \otimes Y$ naturally;
3. $L\partial$ is itself a derivation on $\mathcal{R}$.

Then $1 \iff 2$, and either one implies 3.

$L = X \otimes \_, \partial = \frac{d}{dX}$: think of $L\partial$ as a categorified *Euler homogeneity* operator $f \mapsto X \frac{d}{dX} f$.

Differential operators

Instead study just the diff ops that are polynomials in a given derivative ∂:

- the Arbogast algebra[1] $\mathbf{Arb}[\mathcal{R}, \partial]$ of a D2R $(\mathcal{R}, \partial)$ is the 2-rig generated by $\partial : \mathbf{Cat}_+(\mathcal{R}, \mathcal{R})$

Then, a generic element D of $\mathbf{Arb}[\mathcal{R}, \partial]$ is a finite sum

$$D = \sum_{i \in I} A_i \otimes \partial^{n_i}$$

that can be considered as an endofunctor of $\mathcal{R}$, taking X to $DX = \sum_{i \in I} A_i \otimes (\partial^{n_i} X)$.

A solution for a diffeq prescribed by $D \in \mathbf{Arb}[\mathcal{R}, \partial]$ is a terminal D-coalgebra.

[1] Louis François Antoine Arbogast, (1759–1803) 'λ-abstracted' the notation Df for differential operators $D : C^\infty \to C^\infty$, thereafter thought as *functionals*.

Summing up:

- there's a ring theory to write for 2-rigs
- these objects are highly structured ($\partial I \neq 0$, self-similarity,...)
- it's 'difficult' for a category to be a diff-2-rig ($Der(\mathcal{R})$ knows about a 'dimension' of $\mathcal{R}$)
- yet, differential algebra is quite interesting (differential equations?)

A motivating work in progress

- I got interested in structures like $\mathbf{Mdv}_{\mathcal{R}}$: fix a monoidal cat; $\mathbf{Mdv}_{\mathcal{R}}$ has[2]
    - objects: representations of free monoids $d : A^* \otimes X \to X$
    - morphisms: suitably equivariant maps

Theorem

Let $\mathcal{R}$ be a differential 2-rig. Then, there is a universal monoidal fibration

$$V : \mathbf{Mdv}_{\mathcal{R}} \longrightarrow \mathcal{R}$$

$\mathbf{Mdv}_{\mathcal{R}}$ is a D2R and the functor is a D2R morphism.

[1] 'Medvedev semiautomata'; not that this is important.

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