# Profunctorial Semantics I

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# Algebraic structures

A group is a set equipped with operations

- $m : G \times G \to G$
- $i : G \to G$
- $e : 1 \to G$

...

you know the drill

# Algebraic structures

**Theorem (Higman-Neumann 1953)**
A group is a set equipped with a single binary operation $/ : G \times G \to G$ subject to the single equation

$$x/((((x/x)/y)/z)/(((x/x)/x)/z)) = y$$

for every $x, y, z \in X$.

Well.

This is awkward.

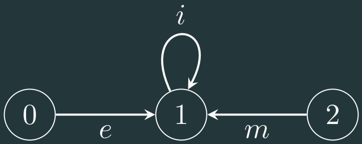
*The theory of equationally definable classes of algebras, initiated by Birkhoff in the early thirties, is [...] hampered in its usefulness by two defects. [...T]he second is the awkwardness inherent in the presentation of an equationally definable class in terms of operations and equations.*

*Quite recently, Lawvere, by introducing the notion - closely akin to the clones P. Hall - of an algebraic theory, rectified the second defect.*

**Definition**
An operator domain is a sequence $\underline{\Omega} = (\Omega_n \mid n \in \mathbb{N})$; the elements of $\Omega_n$ are called operations of arity $n$.

**Definition**
An interpretation $\underline{E}$ of an operator domain $\underline{\Omega}$ consists of a pair $(E, (f_\omega \mid \omega \in \Omega_n, n \in \mathbb{N}))$ where $f_\omega : E^n \to E$ is an $n$-ary operation on the set $E$ called the *carrier* of $\underline{E}$.

An operator domain can be represented as a (rooted) graph: for example, for groups



Way better to use functors.

A **Lawvere theory** is an identity-on-objects functor $p : \mathrm{Fin}^{\mathrm{o}} \to \mathcal{L}$ that commutes with finite products.

Unwinding the definition:

- $\mathcal{L}$ is a category with the same objects as Fin, the category of finite sets and functions;
- $p$ is a functor that acts trivially on objects
- The only thing that can change between Fin and $\mathcal{L}$ is the number of morphisms $[n] \to [m]$.

Equivalently: $p$ is a **promonad** on the opposite of Fin, regarded as an object of the bicategory of profunctors, that preserves the monoidal structure. $\mathcal{L}$ is the Kleisli object of $p$.

$$\left\{\begin{array}{c}\text{identity on obj}\\ \text{left adjoints}\\ p{:}[\mathcal{L},\mathrm{Set}]\to[\mathrm{Fin}^{\mathrm{o}},\mathrm{Set}]\end{array}\right\} \leftrightarrows \left\{\begin{array}{c}\text{monads in Prof}\\ p{:}\mathrm{Fin}^{\mathrm{o}}\rightsquigarrow\mathrm{Fin}^{\mathrm{o}}\end{array}\right\}$$

- The trivial theory is the identity funtor $1_{\mathsf{Fin}} : \mathsf{Fin}^{\mathsf{o}} \to \mathsf{Fin}^{\mathsf{o}}$
- Since $p$ preserves products, it is uniquely determined by its value on $[1]$. This means that if $p : \mathsf{Fin}^{\mathsf{o}} \to \mathcal{L}$ is a Lawvere theory, then every object of $\mathcal{L}$ is $X^n$ if $p[1] = X$.
- The only difference between Fin and $\mathcal{L}$ is thus the set of morphisms $[n] \to [m]$.

The theory of groups is generated by

$$\mathcal{L}_{\mathsf{Grp}} = \begin{tikzcd} {[0]} \arrow[r, "e"'] & {[1]} \arrow[loop above, "i"] & {[2]} \arrow[l, "m"] \end{tikzcd}$$

and their compositions/products.

A model for a Lawvere theory $p$ is a product-preserving functor $\ell : \mathcal{L} \to \text{Set}$.

The category $\text{Mod}(p)$ for a Lawvere theory is a full, reflective subcategory of the category $[\mathcal{L}, \text{Set}]$ of all functors $\mathcal{L} \to \text{Set}$.

**Theorem**
*The following conditions are equivalent ($p : \mathit{Fin}^{o} \to \mathcal{L}$ a theory):*

- $\ell$ *is a model for a Lawvere theory* $\mathcal{L}$*;*
- *The composition* $\ell \circ p : \mathit{Fin}^{o} \to \mathcal{L}$ *preserves finite products;*
- *The composition* $\ell \circ p : \mathit{Fin}^{o} \to \mathcal{L}$ *is* $J$*-representable (with respect to the inclusion* $J : \mathit{Fin} \to \mathit{Set}$*), i.e.*

$$\ell(X[n]) \cong \mathit{Set}(J[n], A)$$

*for some* $A \in \mathit{Set}$*.*

As a consequence of the previous theorem, the square

$$\begin{tikzcd}
\mathsf{Mod}(p) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "\_\circ p"] \\
\mathsf{Set} \arrow[r, "{[J,1]}"'] & {[\mathsf{Fin}^{\mathsf{o}}, \mathsf{Set}]}
\end{tikzcd}$$

is a pullback.

1. $\mathsf{Mod}(p)$ is a reflective subcategory of $[\mathcal{L}, \mathsf{Set}]$. We write $r_! \dashv r$ for the resulting adjunction.
2. The functor $u$ is monadic, with left adjoint $f$.
3. This sets up a functor

$$\mathfrak{M} : \mathsf{Th}_L(\mathsf{Fin}) \to \mathsf{Mnd}_{<\omega}(\mathsf{Set})$$

because the monad $uf$ above is finitary.

# 1. Proof of reflectiveness

The category Mod$(p)$ is reflective:

A functor $F : \mathcal{L} \to$ Set preserves products if and only if it is right orthogonal with respect to all $\sigma_{AB}$ in

$$\begin{tikzcd}
yA \amalg yB \arrow[r, "\forall t"] \arrow[d, "\forall \sigma_{AB}"'] & F \\
y(A \times B). \arrow[ur, dotted] &
\end{tikzcd}$$

Indeed, $F$ is orthogonal to $\sigma_{AB}$ iff $\hom(-, F)$ inverts $\sigma_{AB}$; now consider the chain

$$\begin{aligned}
F(A \times B) &\cong \hom(y(A \times B), F) \\
&\to \hom(yA \amalg yB, F) \\
&\cong \hom(yA, F) \times \hom(yB, F) \\
&\cong FA \times FB
\end{aligned}$$

**Theorem (The small object argument)**
Let $\mathcal{E}$ be a locally presentable category and $\Sigma \subset \hom(\mathcal{E})$ a set of morphism with (finitely) presentable domain; then the subcategory of $\Sigma$-orthogonal object is always reflective and (finitely) accessibly embedded.

**Proof.**
Build a well pointed[1] endofunctor $R : \mathcal{E} \to \mathcal{E}$, with a natural transformation $\eta : X \to RX$; consider

$$\begin{tikzcd} X \arrow[r] & RX \arrow[r] & RRX \arrow[r] & RRRX \arrow[r] & \cdots \end{tikzcd}$$

$R^\infty(X) := \operatorname{colim} R^n(X)$ has a canonical $X \to R^\infty X$, and it is $\Sigma$-orthogonal by construction. It is the desired functor. □

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[1] Well-pointed means that $\eta R^n = R^n \eta$ for all $n$.

# Proof of monadicity

- Monadicity of $u$: a monadic functor has a left adjoint, reflects isomorphisms, and creates $u$-split coequalizers (those parallel pairs that $u$ sends to split coequalizers, have a coequalizer, that $u$ preserves).
  Apart from the existence of $f$, all properties are stable under pullback.
- $u$ commutes with filtered colimits: it is representable by a finitely presentable object.

$$u(\ell) = \ell[1] \cong [\mathcal{L}, \mathsf{Set}](y[1], \ell)$$

# Proof of monadicity

- Being conservative is stable under pullback: conservative functors are a right orthogonal class,[2] precisely $\iota^{\perp}$ where $\iota : \{0 \to 1\} \to \{0 \cong 1\}$. Right orthogonal classes are closed under limits, so under pullbacks.
- Creating coequalizers of $u$-split pairs is stable under pullback:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "s"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "p^*"] \\
\mathcal{C} \arrow[r, "t"'] & \mathcal{L}
\end{tikzcd}$$

  if $p^*$ creates them, so does $u$.
- Every inverse image is monadic.

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[2] If $\mathcal{K}$ is a category, and $\mathcal{S} \subseteq \hom(\mathcal{K})$ a subset of its morphisms, an object is right $\mathcal{S}$-orthogonal if $\hom(-, A)$ inverts every arrow in $\mathcal{S}$.

# $u$ has a left adjoint

The diagram

$$\begin{tikzcd}
\mathsf{Mod}(p) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "\_\circ p"] \\
\mathsf{Set} \arrow[r, "{[J,1]}"'] & {[\mathsf{Fin}^{\mathsf{o}}, \mathsf{Set}]}
\end{tikzcd}$$

of which $u : Mod(p) \to$ Set is a pullback in the 2-category of accessible right adjoints between locally presentable categories; this category has finite limits, thus $u$ is again an accessible right adjoint between locally presentable categories.

# $u$ has a left adjoint

A different construction for the free functor. Every set $A$ defines a unique $\times$-preserving $A^\bullet : \mathsf{Fin}^{o} \to \mathsf{Set} : [n] \mapsto A^n$. The free functor for the theory $p$ acts on objects and morphisms as $\mathrm{Lan}_p A^\bullet$:

$$\begin{tikzcd}
\mathsf{Fin}^{o} \arrow[r, "A^\bullet"] \arrow[d, "p"'] & \mathsf{Set} \\
\mathcal{L} \arrow[ur, dotted, "FA"'] &
\end{tikzcd}$$

$$(FA)[m] \cong \int^{n} A^n \times \mathcal{L}(m, n)$$

This is the composition of left adjoints

$$\begin{tikzcd}
\mathsf{Set} \arrow[r] & {[\mathcal{L}, \mathsf{Set}]} \arrow[r] & Mod(p) \\
A \arrow[r, mapsto] & A \times \mathcal{L}([1], -) \arrow[r, mapsto] & R(A \times \mathcal{L}([1], -))
\end{tikzcd}$$

# $\mathsf{Th}_L(\mathbf{Fin}) \cong \mathsf{Mnd}_{<\omega}(\mathbf{Set})$

Construct a functor in the opposite direction,

$$\mathfrak{Z} : \mathsf{Mnd}_{<\omega}(\mathsf{Set}) \to \mathsf{Th}_L(\mathsf{Fin});$$

given $T$, we consider the composition $\mathsf{Fin} \hookrightarrow \mathsf{Set} \xrightarrow{F^T} \mathsf{Set}^T$ and its bo-ff factorization,

$$\begin{tikzcd}
\mathcal{L}^{o} \arrow[r, "ff"] & \mathsf{Set}^T \\
\mathsf{Fin} \arrow[u, "b"] \arrow[r, "J"'] & \mathsf{Set} \arrow[u, "F^T"']
\end{tikzcd}$$

- the left vertical arrow is a Lawvere theory.
- $\mathsf{Set}^T \cong \mathcal{L}$-models: $$\begin{tikzcd}
\mathsf{Set}^T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "{[b^{o}, \mathsf{Set}]}"] \\
\mathsf{Set} \arrow[r, "{[J,1]}"'] & {[\mathsf{Fin}^{o}, \mathsf{Set}]}
\end{tikzcd} .$$

# Theories as promonads

There is a 2-monad $\tilde{S} : \mathsf{Prof} \to \mathsf{Prof}$ whose algebras are exactly promonoidal categories.

$$\begin{tikzcd}
\mathsf{Prof} \arrow[r, "\tilde{S}"] & \mathsf{Prof} \\
\mathsf{Cat} \arrow[u] \arrow[r, "S"'] & \mathsf{Cat} \arrow[u]
\end{tikzcd}$$

*If $S$ is a monad on $\mathsf{Cat}$ such that the presheaf functor $P : \mathsf{cat} \to \mathsf{Cat}$ lifts to the Eilenberg-Moore category of $S$, then $S$ lifts to the Kleisli category of $P$.*

# Theories as promonads

$S$ is the free monoid monad, so it is defined as

$$S\mathcal{C} = \coprod_{n\geq 0} \mathcal{C}^n$$

Now, an $\tilde{S}$-algebra consists of a 1-cell $\tilde{S}A \rightsquigarrow A$ satisfying certain axioms. We claim that these axioms amount to the request that $A$ is a multicategory of some sort.

First: an $\tilde{S}$-algebra is a multicategory. It is enough to expand the definition as follows: an $\tilde{S}$-algebra is a functor $SA \times A^{\mathrm{o}} \to$ Set, and $SA = \coprod A^n$.

# Theories as promonads

Now, since products distribute over sums, we have

$$\frac{\dfrac{\left(\coprod_{n\geq 0} A^n\right) \times A^{\mathsf{o}} \overset{\otimes_n}{\to} \mathsf{Set}}{\coprod_{n\geq 0}\left(A^n \times A^{\mathsf{o}}\right) \overset{\otimes_n}{\to} \mathsf{Set}}}{\prod_{n\geq 0}\left(A^n \times A^{\mathsf{o}} \overset{\otimes_n}{\to} \mathsf{Set}\right)}$$

This amounts to a family of arrows

$$\otimes_n : A^n \times A^{\mathsf{o}} \longrightarrow \mathsf{Set}$$

such that certain assumptions (associativity and unitality) are satisfied; so $A^{\mathsf{o}}$ is endowed with a multicategory structure, whose set of $n$-ary multimorphisms is exactly $\otimes_n(a_1, \ldots, a_n; a_0)$.

# Theories as promonads

This is, however, a multicategory of a very special kind, where all $(\otimes_n \mid n \geq 3)$ are determined by $\{\otimes_0, \otimes_1, \otimes_2\}$.

To prove this, the associativity axiom for the multiplication comes now into play: every $\otimes_n$ can in fact be determined as a composition of products

$$(w_1 \times \cdots \times w_n) \circ (w_1 \times \cdots \times w_{n-1}) \circ \cdots \circ (w_1 \times w_2) \circ \otimes_2$$

where $w_i$ is either the identity of $A$ or $\otimes_2$ (the associativity axiom implies that all such words are equal to $\otimes_n$).

# Theories as promonads

Given a profunctor $p : \mathcal{A} \rightsquigarrow \mathcal{B}$ between promonoidal categories $(\mathcal{A}, \mathfrak{P}, J_A), (\mathcal{B}, \mathfrak{Q}, J_B)$:

- $p$ is a pseudo-$\tilde{S}$-algebra morphism;
- The cocontinuous left adjoint $\hat{p}$ associated to $p$ is strong monoidal with respect to the convolution monoidal product on presheaf categories;

If $\mathfrak{P}, \mathfrak{Q}$ on $\mathcal{A}, \mathcal{B}$ are representable then

- Both mates $p^{\triangleleft} : \mathcal{A} \to P\mathcal{B}$ che $p^{\triangleright} : \mathcal{B} \to P^*\mathcal{A}$ are strong monoidal wrt convolution on their codomains.

**Theorem**

$$[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set})$$

Proof: use Yoneda lemma.

Just kidding!

The inclusion functor $J : \mathsf{Fin} \to \mathsf{Set}$ extends to

$$\mathrm{Lan}_J : [\mathsf{Fin}, \mathsf{Set}] \to \mathsf{Set} : A \mapsto \int^n Fn \times \mathsf{Set}(Jn, A) = \int^n Fn \times A^n$$

(Yoneda lemma; this time for real). This functor has a right adjoint $J^*$, and $J$ is dense and fully faithful; this entails that $\mathrm{Lan}_J \dashv J^*$ is an equivalence on the subcategory $\mathsf{End}_{<\omega}(\mathsf{Set})$ of finitary functors.

# Theories as $[\mathsf{Fin}, \mathsf{Set}]$-categories

$$[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set})$$

Equivalence is monoidal; the $\circ$-transported structure is called the substitution monoidal product of functors $F, G : \mathsf{Fin} \to \mathsf{Set}$:

$$F \ominus G : m \mapsto \int^{n} Fn \times (Gm)^{n}$$

Substitution is (highly!) non-symmetric, right closed monoidal structure (not left closed).

The category $[\mathsf{Fin}, \mathsf{Set}]$ works as base of enrichment.

From [Garner]

From now on we blur the distinction between the categories $[\mathsf{Fin}, \mathsf{Set}] \cong \mathsf{End}_{<\omega}(\mathsf{Set}) = \mathcal{W}$:

- A finitary monad is a monoid in $\mathcal{W}$, i.e. a $\mathcal{W}$-category with a single object;
- A Lawvere theory is a $\mathcal{W}$-category that is absolute (=Cauchy-, =Karoubi-)complete as an enriched category and generated by a single object.

Lawvere theories form a reflective subcategory in finitary monads; reflection is the enriched Cauchy completion functor.

# Theories as $\mathcal{W}$-categories

In this perspective there is no difference between a Lawvere theory and its associated monad: they are the very same thing, up to a Cauchy-completion operation.

(The Cauchy completion of a monoid in $\mathsf{Cat}$ is rarely a monoid: take the “generic idempotent” $M = \{1, e\}$ and split $e : * \to *$ as $r : 0 \leftrightarrows * : s$).

In order to add all $\mathcal{W}$-absolute colimits, at least all tensors $y[n] \odot X$ must be added to the single object $X$.

# Theories as $\mathcal{W}$-categories

Equivalently,

- A Lawvere $\mathcal{W}$-category is an enriched category where every object $A$ is the tensor $y[n] \odot X$ for a distinguished object $X \cong y[1] \odot X$. All such categories are $\mathcal{W}$-absolute complete.
- A $\mathcal{W}$-category is a special kind of cartesian multicategory: one where a multimorphism $f : X_1 \ldots X_n \to Y$ is such that $X_1 = X_2 = \cdots = X_n$.

Generalisations/extensions:

- let $\mathbb{N}$ be the discrete category over natural numbers;
- let $\mathbf{P}$ be the groupoid of natural numbers;

The categories $[\mathbb{N}, \mathsf{Set}]$ and $[\mathbf{P}, \mathsf{Set}]$ become monoidal with respect to substitution products $\ominus_N, \ominus_P$:

$$F \ominus_N G : n \mapsto \coprod_{k \in \mathbb{N}} G_k \times \coprod_{\vec{n} : \sum n_i = n} X_{n_1} \times \cdots \times X_{n_k}$$

$$F \ominus_P G : n \mapsto \int^{k, \vec{n}} Y_k \times X_{n_1} \times \cdots \times X_{n_k} \times \mathbf{P}\left(\sum n_i, n\right)$$

# PRO(P)S

$\ominus_N$ and $\ominus_P$-monoids are respectively non-symmetric and symmetric operads.

- A PRO is an identity-on-objects strong monoidal functor $p : \mathbb{N}^{o} \to \mathcal{P}$. $\mathcal{P}$ is possibly non-cartesian.
- A PROP is an identity-on-objects strong monoidal functor $p : \mathbb{N}^{o} \to \mathcal{P}$. $\mathcal{P}$ is symmetric monoidal.

Still examples of promonoidal promonads and symmetric promonoidal promonads.

## PRO(P)s and operads

Every PRO $p : \mathbb{N}^{\mathsf{o}} \to \mathcal{T}$ gives rise to the operad $O(\mathcal{T}) = (\mathcal{T}(n,1) \mid n \in \mathbb{N})$.

Conversely, any operad $(\mathcal{O}(n) \mid n \in \mathbb{N})$ gives rise to a pro $T(\mathcal{O})$, where

$$T(\mathcal{O})(n,m) = \coprod_{k_1+\ldots+k_m=n} \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_m).$$

(It would be helpful to imagine a picture of $m$ trees stacked vertically.)

If we begin with an operad $\mathcal{O}$, we have $\mathcal{O} = O(T(\mathcal{O}))$. (This is because $T(\mathcal{O})(n,1) = \mathcal{O}(n)$, according to the above formula.)

On the other hand, if we start with a PRO $\mathcal{T}$, then there exists a canonical map of PROs $T(O(\mathcal{T})) \to \mathcal{T}$, given by, for each $n$ and $m$, a canonical function

$$\coprod_{k_1+\cdots+k_m=n} \mathcal{T}(k_1, 1) \times \cdots \times \mathcal{T}(k_m, 1) \to \mathcal{T}(n, m) \qquad (\star)$$

induced from the monoidal product on $\mathcal{T}$.

This sets up an adjunction

$$T : \mathsf{Opd}[\mathsf{S}] \leftrightarrows \mathsf{PRO}[\mathsf{P}] : O$$

with fully faithful left adjoint, so that [symmetric] operads can be regarded as a PRO[P]s $\mathcal{T}$ such that each function (*) is bijective.

# The evil plan

Re-enact [Garner] away from Set.

Let $\mathcal{V}$ be a locally presentable base of enrichment; let $\mathfrak{F}(\mathcal{V})$ be the subcategory of finitely presentable objects:

- $\mathfrak{F}(\mathcal{V})$ is the free finite weighted cocompletion of the point;
- There is a strong monoidal equivalence of categories

  $$[\mathfrak{F}(\mathcal{V}), \mathcal{V}] \cong [\mathcal{V}, \mathcal{V}]_{<\omega}$$

  between functors $\mathfrak{F}(\mathcal{V}) \to \mathcal{V}$ and finitary endo-$\mathcal{V}$-functors;

# The evil plan

- $\mathcal{V}$-substitution is

$$F * G = A \mapsto \int^{B} FB \otimes_{\mathcal{V}} (GA)^{B} \leftarrow \mathcal{V}\text{-power}$$

- Equivalence between finitary $\mathcal{V}$-monads and enriched-Cauchy-complete categories generated by a single object under iterated finite powers.
- Models for a Lawvere theory correspond to algebras for the associated finitary monad; free models are free agebras are representables in

$$\begin{aligned} \mathsf{Alg}(T, \mathcal{C}) &= [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-}\mathsf{Cat}(T, \mathcal{C}) \\ (\text{Cauchy compl.}) &\cong [\mathfrak{F}(\mathcal{V}), \mathcal{V}]\text{-}\mathsf{Cat}(\hat{T}, \mathcal{C}) \\ &= \mathsf{Mod}(\hat{T}, \mathcal{C}) \end{aligned}$$

# The evil plan

| class of lims | finite × | $\mathbb{D}$-limts | finite powers | weighted $\mathbb{D}$-limits | bicat × |
|---|---|---|---|---|---|
| basic theory | Fin$^{o}$ | completion of $\{*\}$ | completion of $\{*\}$ | completion of $\{*\}$ | completion of $\{*\}$ |
| semantics in | Set | Set | $\mathcal{V}$ | $\mathcal{V}$ | Prof |
| eq. with _ monads | finitary | $\mathbb{D}$-accessible | $[\mathfrak{F}(V), V]$-monoids | $[?, V]$-monoids | ??? |

# Profunctorial semantics

- Characterise the free carbicat $\mathbb{CB}(*)$ on a singleton: see link here);
- Check if the univ property of Fin remains true for $\mathbb{CB}(*)$;
- Take $\mathbb{CB}(*) = F$, and consider its free cocompletion in the bicolimit sense
- Prove that

$$\begin{aligned} [PF, PF] &\cong [\mathbb{CB}(*), PF] \\ &\cong PF \end{aligned}$$

monoidally; $\odot$-monoids := monoids in $PF$ wrt composition in $[PF, PF]$.

# Profunctorial semantics

- Prove that there is a syntax-VS-semantics adjunction here: theories are promonoidal promonads $T$ on (a 1-skeleton of) $\mathbb{CB}(*)$, and models are carbicat homomorphisms $\mathsf{Kl}(T) \to \mathsf{Prof}$. There is an equivalence

$$\{\text{theories}\} \cong \{\text{??? monads}\}$$

- Let PROs come into play: analogue of the adjunction between PROs and operads.

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