From combinatorial species to general differential 2-rigs

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Goals : in the context of general differential 2-rigs (Loregian, Trimble, [5]) :

- Can we solve differential equations using the same techniques as for combinatorial species ?
- Can some theorems about combinatorial species be extended ?

Summary

Definition, Loregian, [5].

A 2-rig is a category $\mathcal{C}$ with :

- finite coproducts $+$, called the addition,
- an other monoidal structure $\otimes$, called the multiplication,
- natural isomorphisms :

$$X \otimes Y + X \otimes Z \underset{\sim}{\xrightarrow{\delta^L}} X \otimes (Y + Z)$$

$$Y \otimes X + Z \otimes X \underset{\sim}{\xrightarrow{\delta^R}} (Y + Z) \otimes X$$

Example

- $(\mathrm{Set}, +, \times, 1)$.
- If R is a ring, then $(Mod_R, \oplus, \otimes, R)$ is an example.
- If $(\mathcal{A}, \oplus, j)$ is a monoidal category, then $([\mathcal{A}^{op}, \mathrm{Set}], +, *, I = \mathcal{A}(j, -))$ is an example, where $*$ is the Day convolution :

$$F * G = \int^{U,V \in \mathcal{A}} FU \times GV \times \mathcal{A}(U \oplus V, -)$$

- If $\mathcal{C}$ is a 2-rig, then the category $\mathcal{C}[Y]$ with objects finite families of objects of $\mathcal{C}$ noted $(A_1, \ldots, A_n) = \sum_{i=0}^{n} A_i \otimes Y^i$ with component-wise sum and Cauchy product :

$$\left(\sum_{i=0}^{n} A_i \otimes Y^i\right) \otimes \left(\sum_{j=0}^{m} B_j \otimes Y^j\right) = \left(\sum_{k=0}^{m+n} \left(\sum_{i+j=k} A_i \otimes B_j\right) \otimes Y^k\right)$$

Definition, Loregian, [5].

A differential 2-rig is a 2-rig $\mathcal{C}$ with :

- an endofunctor ∂, called the derivation,
- natural isomorphisms :

$$\partial X + \partial Y \underset{\sim}{\xrightarrow{\partial i_X + \partial i_Y}} \partial(X + Y)$$

$$\partial X \otimes Y + X \otimes \partial Y \underset{\sim}{\xrightarrow{l}} \partial(X \otimes Y)$$

such that : naturality, compatibility with the left-/right-distributors, compatibility with the $\otimes$-associator, compatibility with the left-/right-$\otimes$-unitors.

Ex for naturality : for all morphisms $u : X \to X'$, $v : Y \to Y'$, we want the following diagram to commute :

$$\begin{tikzcd}
\partial(X \otimes Y) \arrow[r, "\partial(u \otimes v)"] & \partial(X' \otimes Y') \\
\partial X \otimes Y + X \otimes \partial Y \arrow[u, "l_{X,Y}"] \arrow[r, "\partial u \otimes v + u \otimes \partial v"'] & \partial X' \otimes Y' + X' \otimes \partial Y' \arrow[u, "l_{X',Y'}"']
\end{tikzcd}$$

Example

- If we want to endow $(\mathcal{C}[Y], +, \otimes, I)$ with a derivation satisfying $\partial Y = I$, the Leibniz rule impose to set :

$$\partial \sum_{i=0}^{n} A_i \otimes Y^i = \sum_{i=0}^{n-1} (i+1) A_{i+1} \otimes Y^i$$

where $(i+1)A_{i+1}$ is the sum of $(i+1)$ copies of A_{i+1}.

Definition 1, Joyal, [3].

Let $\mathcal{B}$ be the category of finite sets, with morphisms being the bijections. Define the category of combinatorial species $\mathrm{Spc} = [\mathcal{B}, \mathrm{FinSet}]$, which is equivalent to $[\mathcal{B}, \mathcal{B}]$.

Remark.

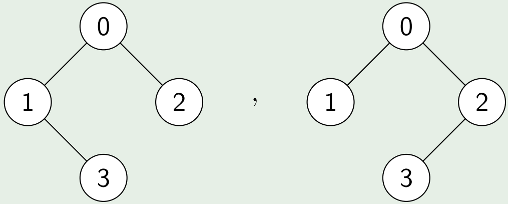
Decompose : $\mathcal{B} \simeq \coprod_{n=0}^{\infty} S_n$
So $X : \mathcal{B} \to \mathrm{FinSet}$ can be decomposed as :

- a sequence of finite sets X_n, $n \geq 0$,
- a sequence of left actions of S_n on X_n, $n \geq 0$.

Example (species of trees)

Define the species of trees, by assigning to a finite set E the set of trees on E :

$4 \mapsto$



$, \cdots$

and the action of S_n on X_n permutes the vertices of a tree chosen in the set X_n.

Structure of differential 2-rig on Spc : for species X, Y and a finite set E :

- Sum :

$$(X+Y)(E) = X(E) + Y(E) = X(E) \coprod Y(E)$$

- Multiplication :

$$(X \otimes Y)(E) = \sum_{E_1+E_2=E} X(E_1) \times Y(E_2)$$

- Derivation :

$$(\partial X)(E) = X(E+1) = X(E+\{*\})$$

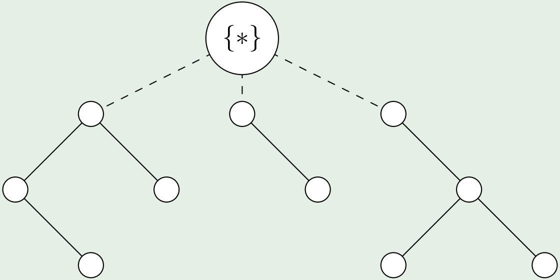
Additional structure on Spc : for species X, Y and a finite set E :

- Substitution :

$$(X \circ Y)(E) = \sum_{\pi \text{ partition of } E} X(\pi) \times \prod_{p \in \pi} Y(p)$$

Example (derivative of the species of trees, Bergeron, [2])

If X is the species of trees, the species ∂X assigns to a finite set E the set of trees on $E + \{*\}$:



So ∂X is the species of disjoint sets of rooted trees.

Summary

First goal : can we solve (some) differential equations in general 2-rigs ?
Polynomial differential equations : finding fixed points of :

$$X \mapsto A_0 + A_1 \otimes \partial X + A_2 \otimes (\partial^2)X + \cdots + A_n \otimes (\partial^n)X$$

For instance :

$$X \mapsto \partial X$$

Technique : use initial algebras and terminal coalgebras to find fixed points of functors.

Example

Take a set A. What are the fixed points of the following functor ?

$$\begin{array}{rcl} T_A : \mathrm{Set} & \to & \mathrm{Set} \\ S & \mapsto & 1 + (A \times S) \end{array}$$

Start from the initial object $\varnothing$ or the terminal object 1, and recursively apply T_A to the unique morphisms $\varnothing \xrightarrow{!_1} T_A(\varnothing)$ and $1 \xleftarrow{!_2} T_A(1)$:

Example

$$\varnothing \xrightarrow{!_1} 1 \xrightarrow{T_A !_1} 1 + A \xrightarrow{T_A^2 !_1} 1 + A + A^2 \xrightarrow{T_A^3 !_1} 1 + A + A^2 + A^3 \to \dots$$

$$1 \xleftarrow{!_2} 1 + A \xleftarrow{T_A !_2} 1 + A + A^2 \xleftarrow{T_A^2 !_2} 1 + A + A^2 + A^3 \leftarrow \dots$$

Taking :

- the colimit of the first equation gives A^*, ie the initial algebra of T_A,
- the limit of the second equation gives $A^* + A^{\mathbb{N}}$, ie the terminal coalgebra of T_A,

and they give solutions to $T_A(X) \simeq X$.

Theorem, Trnoková et al.

A set functor has an initial algebra if and only if it has a fixed point.

First Adámek's theorem, [6].

If $\mathcal{C}$ has an initial object 0 and ω-composition, and $F : \mathcal{C} \to \mathcal{C}$ preserves colimits of ω-chains, then the initial algebra of F is the colimit of :

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \to \dots$$

Second Adámek's theorem, [1].

If $\mathcal{C}$ has colimits and $F : \mathcal{C} \to \mathcal{C}$ preserves colimits of λ-chains for some infinite ordinal λ, then the initial algebra of F is $F^\lambda 0 \xrightarrow{F^\lambda !} F^{\lambda+1} 0$.

Lambek's theorem.

If $F : \mathcal{C} \to \mathcal{C}$ has an initial algebra $\alpha : F(X) \to X$, then α is an isomorphism.

Remark.

Dual versions also work.

Remark.

If they exist :

- the initial algebra is the smallest fixed point,
- the terminal coalgebra is the largest fixed point.

Difficult :

- comodules : no
- linear species $([GL(p), Vect_k], \oplus, \otimes)$: no
- etc.

Idea : $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$ are monoidal categories.

Structure on $[(\mathbb{N},+), Vect_k]$.

Consider $[(\mathbb{N},+), Vect_k]$ with 'Day convolution'. That is, for objects F, G:

$$F+G=(F_n\oplus G_n)_{n\in\mathbb{N}}$$

$$\begin{aligned} F*G &= \left(\int^{p,q\in\mathbb{N}}(F(p)\otimes G(q))\odot\mathbb{N}(p+q,n)\right)_{n\in\mathbb{N}} \\ &= \left(\sum_{p+q=n}F(p)\otimes G(q)\right)_{n\in\mathbb{N}} \end{aligned}$$

$$I=(k,0,0,\dots)$$

Derivation ? Copy polynomials :

$$\partial F=((n+1)F_{n+1})_{n\in\mathbb{N}}=\left(\bigoplus_{1\leq k\leq n+1}F_{n+1}\right)_{n\in\mathbb{N}}$$

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$$I = (0, k, 0, 0, \dots)$$

Derivation ? For a prime number r :

$$\partial F = \partial_r F = 0 \oplus (\delta_n F_{r\cdot n})_{n\geq 1}$$

for some coefficients δ_n. Only choice of coefficients :

$$\partial F = \left(0, ((v_r(n)+1)F_{r\cdot n})_{n\geq 1}\right)$$

Can we use the initial algebra or coalgebra techniques to solve the differential equation $\partial V \simeq V$ in our two examples of structures ?

$0 = (0, 0, \dots)$ is both initial and terminal. We want to study :

$$0 \xrightarrow{!} \partial 0 \xrightarrow{\partial !} \partial^2 0 \to \dots$$

$$0 \xleftarrow{!} \partial 0 \xleftarrow{\partial !} \partial^2 0 \leftarrow \dots$$

Issue : in our two structures we have $\partial 0 = 0$.
We even have $\partial I = 0$.

Let's completely solve the differential equation $\partial V \simeq V$ in our two examples of structures.

Solutions in $[(\mathbb{N}, +), Vect_k]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the $\mathbb{N}$-graded vector spaces of the form $V = (k^\alpha)_{n \geq 0}$ for an infinite cardinal α, and the trivial space.

Proof.

$$\begin{aligned} \partial V \simeq V \quad & \Leftrightarrow \quad \forall n,\ V_n \simeq (n+1) V_{n+1} \\ & \Rightarrow \quad V_0 \simeq V_1 \simeq 2 V_2 \simeq 3! V_3 \simeq \cdots \simeq n! V_n \simeq \dots \end{aligned}$$

3 steps :

- except the trivial solution, the dimensions must be infinite,
- assume $V = (k^{\alpha_n})_n$,
- equation on the dimensions α_n :

$$\forall n,\ \alpha_n \simeq (n+1)\alpha_{n+1}$$

Remark.

Imposing $V_0 = \Lambda$ for some infinite dimensional vector space Λ, we get exactly one solution up to isomorphism :

$$V = (\Lambda, \Lambda, \dots)$$

Remark.

If Λ is a non-trivial finite dimensional vector space, there is no solution.

Is $V_0 = \Lambda$ a nice initial condition ? Like $X[\varnothing] = \varnothing$ for species used by Labelle in [4], in

$$\begin{cases} \partial X = X \\ X[\varnothing] = \varnothing \end{cases}$$

Remark.

Similarly we can solve :

$$\begin{cases} \partial V \simeq A \otimes V + B \\ V_0 = \Lambda \end{cases}$$

but only under some conditions on A, B, Λ.

Definition.

For $n \in \mathbb{N}$, write the decomposition

$$n = w_r(n) r^{v_r(n)}$$

Solutions in $[(\mathbb{N}, \cdot), Vect_k]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the $\mathbb{N}$-graded vector spaces of the form $V = \big(0, (U_{w_r(n)})_{n \geq 1}\big)$, where, for w prime to r, U_w is the trivial space or of the form k^{α_w} for an infinite cardinal α_w.

Proof.

$$\partial V \simeq V \quad \Leftrightarrow \quad V_0 = 0 \quad \text{and} \quad \forall n \geq 1,\ V_n \simeq (v_r(n)+1)V_{rn}$$

$$\Leftrightarrow \quad V_0 = 0 \text{ and } \forall w \text{ prime to } r, \forall v \geq 0,\ V_{wr^v} \simeq (v+1)V_{wr^{v+1}}$$

$$\Leftrightarrow \begin{cases} V_0 = 0 \\ V_1 \simeq V_r, & V_r \simeq 2V_{r^2}, & V_{r^2} \simeq 3V_{r^3}, & V_{r^3} \simeq 4V_{r^4} & \dots \\ V_2 \simeq V_{2r}, & V_{2r} \simeq 2V_{2r^2}, & V_{2r^2} \simeq 3V_{2r^3}, & V_{2r^3} \simeq 4V_{2r^4} & \dots \\ V_3 \simeq V_{3r}, & V_{3r} \simeq 2V_{3r^2}, & V_{3r^2} \simeq 3V_{3r^3}, & V_{3r^3} \simeq 4V_{3r^4} & \dots \\ \dots \\ V_w \simeq V_{wr}, & V_{wr} \simeq 2V_{wr^2}, & V_{wr^2} \simeq 3V_{wr^3}, & V_{wr^3} \simeq 4V_{wr^4} & \dots \\ \dots \end{cases}$$

Set $U_v^{(w)} = V_{wr^v}$ for w prime to r, and use the fact that each $n \in \mathbb{N}$ has a unique decomposition $n = wr^v$ with w prime to r.

Remark.

Imposing $V_w = \Lambda^{(w)}$ for some infinite dimensional vector spaces $\Lambda^{(w)}$ for w prime to r, we get exactly one solution up to isomorphism.

Is $V_w = \Lambda^{(w)}$ for w prime to r a nice initial condition ?

Summary

Definition 2.1, Labelle [4].

Given species $F_{i,j}$, a solution of the differential problem

$$\left\{\begin{array}{rcll} \partial Y_i & = & F_{i,j}(X_1,\dots,X_k,Y_1,\dots,Y_p), & 1\le i\le p, 1\le j\le k \\ Y_i[\varnothing,\dots,\varnothing] & = & \varnothing, & 1\le i\le p \end{array}\right.$$

is a family of species $A=(A_i(X_1,\dots,X_k))_{1\le i\le p}$ and natural isomorphisms

$$\theta_{i,j} : \partial A_i/\partial X_j \xrightarrow{\sim} F_{i,j}(X_1,\dots,X_k,A_1,\dots,A_p)$$

such that

$$A_i[\varnothing,\dots,\varnothing]=\varnothing, \quad 1\le i\le p$$

Example

$$\left\{\begin{array}{rcl} \partial X & = & A\otimes X+B \\ X[\varnothing] & = & \varnothing \end{array}\right.$$

Part of theorem A, Labelle [4].

If m is a finite (possibly null) cardinal number or $m = 2^{\aleph_0}$, then there exists a normalized compatible differential problem having exactly m non-isomorphic combinatorial solutions. Moreover, no differential problem can have exactly $m = \aleph_0$ or $m > 2^{\aleph_0}$ non-isomorphic combinatorial solutions.

Lemma 2.6, Labelle [4].

For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, there exists only a finite number $\mu_n > 0$ of non-isomorphic molecular species

$$M_n^{(i)} = M_n^{(i)}(X_1, \dots, X_k))$$

supported by multisets having multicardinality n.
Every species $H = H(X_1, \dots, X_k)$ has a unique molecular decomposition of the form

$$H = \sum_{n \in \mathbb{N}^k,\ 1 \le i \le \mu_n} C_n^{(i)}(H) M_n^{(i)}$$

where $C_n^{(i)}(H)$ are natural integers.
Moreover, for any pair H, K of species we have

$$H \simeq K \quad \Leftrightarrow \quad \forall n, \forall i, C_n^{(i)}(H) = C_n^{(i)}(K)$$

Conjecture.

If $\mathcal{C}$ is a monoidal category with initial object 0, such that the cardinality of $\mathcal{C}_0$ is κ, and such that the 2-rig $[\mathcal{C}^{op}, \mathrm{Set}]$ can be endowed with a derivation ∂, then the differential problem :

$$\begin{cases} \partial X & \simeq & X \\ X[0] & = & \{*\} \end{cases}$$

has at most 2^{κ} solutions.

We want to replace $[(\mathbb{N},+), Vect_k]$ with something of the form $[\mathcal{C}^{op}, \mathrm{Set}]$
:

- Replace $(\mathbb{N},+)$ by $(\mathbb{N},\geq,\min) = (\mathbb{N},\leq,\max)^{op}$.
- We want to replace $Vect_k$ by Set : same properties :

$$k^\alpha \oplus k^\beta = k^{\alpha+\beta}$$

$$k^\alpha \otimes k^\beta = k^{\alpha\times\beta}$$

Define the differential 2-rig $[(\mathbb{N}, \geq, \min), \mathrm{Set}]$:

- Sum :

$$F + G = (F_n + G_n)_{n\in\mathbb{N}}$$

- Multiplication :

$$\begin{aligned} F * G &= \left(\int^{p,q\in\mathbb{N}} F(p) \times G(q) \times \mathbb{N}(n, \min(p,q))\right)_{n\in\mathbb{N}} \\ &= \left(\sum_{n\leq p,q} F(p) \times G(q)\right)_{n\in\mathbb{N}} \end{aligned}$$

- Derivation :

$$\partial F = \left(\coprod_{k\in\aleph_0} F_n\right)_{n\in\mathbb{N}} = (\aleph_0 F_n)_{n\in\mathbb{N}}$$

Is ∂ really Leibniz ? For example for naturality.
On objects F, G, at the level $n \geq 0$:

$$\left\{\begin{array}{rcl} (\partial(F * G))_n & = & \coprod_{k \in \aleph_0} \coprod_{n \leq p,q} F(p) \times G(q) \\ (\partial F * G + F * \partial G)_n & = & \coprod_{n \leq p,q} \left(\coprod_{k \in \aleph_0} F(p)\right) \times G(q) \\ & & + \coprod_{n \leq p,q} F(p) \times \left(\coprod_{k \in \aleph_0} G(q)\right) \\ & \simeq & \coprod_{t \in \{0,1\}} \coprod_{k \in \aleph_0} \coprod_{n \leq p,q} F(p) \times G(q) \end{array}\right.$$

The above isomorphism is natural. If we fix a bijection $\aleph_0 \simeq \{0,1\} \times \aleph_0$, independently of F, G, we can show we have a natural isomorphism between the two above expressions,by reindexing.

Goal : solve $\partial V \simeq V$ in this structure.

Solutions in $[(\mathbb{N}, \geq, \min), \mathrm{Set}]$.

The solutions of $\partial V \simeq V$ are, up to isomorphism, the objects $V = (V_n)_{n\in\mathbb{N}}$ such that each V_n is an infinite set or 0.

Proof.

$$\partial V \simeq V \quad \Leftrightarrow \quad \forall n \geq 0,\ V_n \simeq \aleph_0 V_n$$

So $V_0 = 0$ or even $V_0 = \Lambda$ doesn't fix a 'reasonable' number of solutions : $(\mathbb{N}, \geq, \min)$ has $\aleph_0$ objects, but we have strictly more than $2^{\aleph_0}$ solutions even with the initial condition.

Summary

Recall Labelle's decomposition of combinatorial species :

$$H = \sum_{n \in \mathbb{N}^k,\ 1 \leq i \leq \mu_n} C_n^{(i)}(H) M_n^{(i)}$$

where $C_n^{(i)}$ are natural integers and $M_n^{(i)}$ are molecular species.

If we :

- allow negative coefficients, writing $H = H_p - H_n$ for two species H_p, H_n,
- quotient up to $H_p - H_n = H'_p - H'_n \Leftrightarrow H_p + H'_n \simeq H'_p + H_n$,

we get the virtual species.

It can give solutions to equations which otherwise wouldn't have any.

Definition.

A category $\mathcal{C}$ is cancellative if for every objects A, B, C, the property $A + B \simeq A + C$ implies $B \simeq C$.

Consider a cancellative differential 2-rig $(\mathcal{C}, +, \otimes, \partial)$.

Definition.

Set $(\mathcal{C}^2, \boxplus, \boxtimes, \bar{\partial})$, where :

$$(A, B) \boxplus (C, D) = (A + C, B + D)$$

$$(A, B) \boxtimes (C, D) = (A \otimes C + B \otimes D, A \otimes D + B \otimes C)$$

$$\bar{\partial}(A, B) = (\partial A, \partial B)$$

Theorem.

$(\mathcal{C}^2, \boxplus, \boxtimes, \bar{\partial})$ is a differential 2-rig.

Definition.

The virtual category $\mathbb{V}(\mathcal{C})$ is $\mathcal{C}^2$ quotiented by $(A, B) \sim (C, D)$ if and only if $A + D \simeq C + B$, ie the category with :
- objects : $\mathcal{C}_0^2$ quotiented by $\sim$,
- morphisms $[(A, B)] \to [(C, D)]$: the morphisms $(A', B') \to (C', D')$ for all $(A, B) \sim (A', B')$ and $(C, D) \sim (C', D')$.

Theorem.

The virtual category $\mathbb{V}(\mathcal{C})$ is a differential 2-rig.

Theorem.

$\mathcal{C}$ quotiented by isomorphisms, can be embedded into $\mathbb{V}(\mathcal{C})$ as a differential 2-rig.

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