The fibration of Lyapunov functions

A categorist's take on
the Fundamental Theorem of Dynamical Systems

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A joint work with

(Still many open threads, be patient!)

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Outline

Introduction

Dynamical Systems

A **topological dynamical system** is a pair (X, f), where X is a compact metric space and $f\colon X \to X$ is a continuous (or more strictly, nonexpansive) self-map.

- A **discrete** system is defined by iterations f^n, where $f^0 = \mathrm{id}_X$ and $f^{n+1} = f \circ f^n$
- A **continuous** system is a continuous mapping $\Phi\colon \mathbb{R} \times X \to X$ satisfying $\Phi(0, x) = x$ and $\Phi(t, \Phi(s, x)) = \Phi(t + s, x)$.

Remark

Both are **representations** of a monoid, i.e. functors out of the monoid, into the category **Met** of metric spaces and nonexpansive maps, or metrizable topological spaces and continuous maps.

$$\mathbf{MTop}^{\mathbb{N}} \qquad \mathbf{Met}^{\mathbb{N}} \qquad \mathbf{MTop}^{\mathbb{R}} \qquad \mathbf{Met}^{\mathbb{R}}$$

Recent work of Das-Suda and Vandervorst takes advantage of the natural topological/metric enrichment of these categories (more info upon request).

Dynamical systems

- **Orbits**: The forward orbit of a point x is

$$O^+(x) = \{f^n(x) : n \geq 0\}.$$

- **Invariance**: A subset $S \subseteq X$ is **invariant** if $f(S) \subseteq S$ (or $f(S) = S$ for homeomorphisms).

Remark

To a dynamical system (X, f) one can associate the **action category** $X /\!\!/ f$, whose objects are the points of X and morphisms $n : x \to y$ are given by n such that $f^n(x) = y$.

An orbit is a connected component of $X /\!\!/ f$; an invariant set is a subobject of (X, f) in $\mathbf{Met}^{\mathbb{N}}$, or equivalently, a subcategory of $X /\!\!/ f$.

The Chain Recurrence Relation

- **ϵ-Chains**: For $\epsilon > 0$, an **ϵ-chain** of length n from x to y is a finite sequence

$$C_n : a_0, a_1, \dots, a_n$$

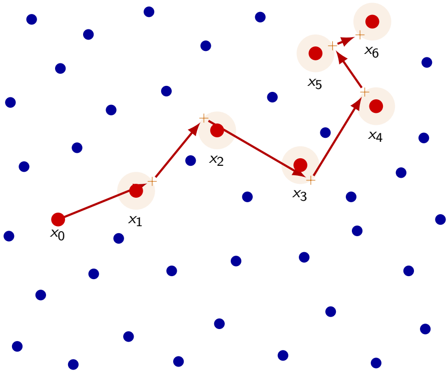
  such that $a_0 = x$, $a_n = y$, and $\max_i d(f(a_i), a_{i+1}) < \epsilon$.

- **Chain Relation**: Write $x \,\mathcal{C}\, y$ if for every $\epsilon > 0$, there exists an ϵ-chain from x to y.

Definition

Let $U_{\epsilon,f} := \{(x, y) \mid d(fx, y) < \epsilon\}$; then let $U^+_{\epsilon,f} := \bigcup_{n\geq 1} U^n_{\epsilon,f}$ be the transitive closure of $U_{\epsilon,f}$; then

$$x \,\mathcal{C}\, y \iff (x, y) \in \bigcap_{\epsilon>0} U^+_{\epsilon,f}.$$



$f : x_n \mapsto \widetilde{x}_{n+1},\ \|\widetilde{x}_{n+1} - x_{n+1}\| \leq \varepsilon$

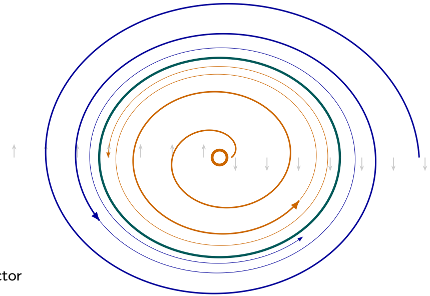
ε-ball around x_n

The Chain Recurrence Relation

- **Chain Recurrence**: A point x is **chain recurrent** if $x \mathcal{C} x$. The set of all chain recurrent points is denoted $Rec(X, f)$; on $Rec(X, f)$, $\mathcal{C}$ is a closed equivalence relation (equivalently, a closed preorder on X whose symmetric closure is the chain-equivalence relation).
- **Poincaré Recurrence**: if X is a compact probability space and $f : X \to X$ is measure-preserving, then the set $Rec(X, f)$ has measure 1.
- Every **orbit** is a chain component, but many points that are not in the same orbit can be chain related (e.g. points in the same attractor basin).
- Chain recurrence is **independent** of the choice of the metric that realizes the same topology.

Conley Theory

- Conley theory provides a global decomposition of the state space into **recurrent** parts ($Rec(X, f)$) and **transient** parts where the system flows in a single direction.
- An **attractor** A is a compact invariant set that is the ω-limit set of a neighborhood of itself. Its dual **repeller** A^* consists of all points whose ω-limit set is disjoint from A.



Van derPol's attractor

Conley Theory

Let $f : X \to X$ be a homeomorphism of a compact space X, $A \subset X$ compact;

- A is called an **attractor** if there exists an open neighborhood $U \supset A$ such that $f(\overline{U}) \subset U$ and $A = \bigcap_{n \geq 0} f^n(\overline{U})$. Then such a U is an **isolating neighborhood**.

Setting $V = X \setminus \overline{U}$ and $A^* = \bigcap_{n \geq 0} f^{-n}(\overline{V})$:

- A^* is an attractor for f^{-1} with isolating neighborhood V
- A^* is called the *repeller dual* to A
- A^* is independent of the choice of U
- $f(A) = A$ and $f(A^*) = A^*$

Theorem (Conley)

The chain recurrent set is the intersection over all attractor-repeller pairs:

$$Rec(X, f) = \bigcap \{A \cup A^* : (A, A^*) \text{ is an attractor-repeller pair}\}.$$

In the compact case, the set of attractors for f is countable.

Conley theory

The coreflection of recurrent points

- On $Rec(X, f)$, the relation

$$x \overset{\mathcal{C}}{\sim} y$$

  defined by "$x\, \mathcal{C}\, y$ and $y\, \mathcal{C}\, x$" is an **equivalence relation**.
- The equivalence classes are called **chain components** or **basic sets**.
- The set of components $CQ(X, f)$ forms a **compact, metrizable, and totally disconnected** space.

The correspondence

$$\begin{aligned} CQ : \mathbf{Met}^{\mathbb{N}} &\longrightarrow \mathbf{Met} \\ (X, f) &\longmapsto Rec(X, f)/_{\overset{\mathcal{C}}{\sim}} \end{aligned}$$

is a functor (the **C**onley **Q**uotient functor).

The coreflection of recurrent points

CQ splits as the composite of two functors; both are universal in a sense.

$$\mathbf{Met} \underset{(-)/_{\underset{\sim}{\mathcal{C}}}}{\overset{i}{\underset{\top}{\rightleftarrows}}} \mathbf{RecMet}^{\mathbb{N}} \underset{R}{\overset{j}{\underset{\bot}{\rightleftarrows}}} \mathbf{Met}^{\mathbb{N}}$$

- $j \circ i$ is the inclusion of metric spaces into dynamical systems, as discrete metric spaces; the inclusion factors through the subcategory of **full-recurrent** metric spaces (those where $Rec(X, f) = X$). CQ is the composite of the quotient by $\mathcal{C}$ and of R. But
- j has a right adjoints; i has a left adjoint.
- Sending a dynamical system to its set of recurrent points is a **mono-coreflection** with counit the inclusion $Rec(X, f) \subseteq X$. Thus, R is the right adjoint to j.
- On full-recurrent spaces, the quotient $(Y, g) \mapsto (Y, g)/_{\underset{\sim}{\mathcal{C}}}$ is a left adjoint to i.

Recall:
To a dynamical system (X, f) one can associate the **action category** $X /\!\!/ f$, whose objects are the points of X and morphisms $x \to y$ are given by the existence of an n such that $f^n(x) = y$.

An orbit is a connected component of $X /\!\!/ f$; an invariant set is a subobject of (X, f) in $\mathbf{Met}^{\mathbb{N}}$, or equivalently, a subcategory of $X /\!\!/ f$.

Contrary to the situation with the orbit quotient, the Conley quotient **is not a left adjoint**;
but it gets close: $CQ : (X, f) \mapsto Rec(X, f)/_{\underset{\sim}{\mathcal{C}}}$ is the composition of a **left and a right** adjoint. Yet,

Remark

CQ does not preserve all coequalizers; it can't be a left adjoint.

Lyapunov functions

Complete Lyapunov Functions

A **complete Lyapunov function** is a continuous map

$$L\colon X \to \mathbb{R}$$

subject to the following properties:

- CLF1 (**monotonicity**) on the complement of $Rec(X, f)$, $L(fx) < L(x)$; and $L(fx) = Lx$ **if and only if** $x \in Rec(X, f)$;
- CLF2 (**orbit-injectivity**) L is constant on the chain components, and it descends to an **injective** map from the quotient;
- CLF3 (**meagreness**) the image of L is a compact nowhere dense subset of $\mathbb{R}$ (of Cantor type).

Theorem (Conley)

For any continuous system (X, f) on a compact metric space, a complete Lyapunov function exists.

Orbit-Injective Maps

Definition

A morphism $u : (X, f) \to (Y, g)$ in $\mathbf{Met}^{\mathbb{N}}$ is **orbit-injective** (or *chain-orbit-injective*) if the induced map between Conley quotients

$$\bar{u} : CQ(X, f) \longrightarrow CQ(Y, g)$$

is *injective*.

Remark

Every complete Lyapunov function $L : (X, f) \to (\mathbb{R}, \mathrm{id})$ is orbit-injective (by condition CLF2).

The Category **CLF**

Definition

The category **CLF** has:

- *Objects*: pairs $((X, f), V)$ where V is a complete Lyapunov function for (X, f).
- *Morphisms* $h : ((X, f), V) \to ((Y, g), W)$: non-expansive equivariant maps $h : X \to Y$ such that

$$V(x) \leq W(h(x)) \quad \text{for all } x \in X.$$

There is an obvious *forgetful functor*

$$P : \mathbf{CLF} \longrightarrow \mathbf{Met}^{\mathbb{N}}, \qquad ((X, f), V) \mapsto (X, f).$$

Posetal Fibrations

Definition

A functor $p : \mathcal{E} \to \mathcal{B}$ is a **posetal fibration** if:

PF1. Each *fibre* $\mathcal{E}_B := \{f \in \hom(\mathcal{E}) \mid pf = \mathrm{id}_B\}$ is a partially ordered set.

PF2. Every arrow $u : B \to B'$ in $\mathcal{B}$ induces a *monotone* reindexing map $u^* : \mathcal{E}_{B'} \to \mathcal{E}_B$ (note the reversal of direction).

Definition (M-fibration)

Let $M \subseteq \mathcal{B}$ be a wide subcategory. p is a *posetal M-fibration* if PF1 holds and PF2 holds for all $u \in M$ only.

In our case: $M = \mathbf{OInj} \subseteq \mathbf{Met}^{\mathbb{N}}$.

The Fibration of Lyapunov Functions

Theorem

The forgetful functor $P : \mathbf{CLF} \to \mathbf{Met}^{\mathbb{N}}$ is a **posetal OInj-fibration**.

Proof.

Orbit-injectivity of $u : (X, f) \to (Y, g)$, is needed to show that the the reindexing

$$u^* : \mathbf{CLF}_{(Y,g)} \longrightarrow \mathbf{CLF}_{(X,f)}, \qquad W \mapsto W \circ u$$

is well-defined.

- CLF1 follows from equivariance of u.
- CLF2 uses *orbit-injectivity* of u (essential here: $W(u(x)) = W(u(y)) \Leftrightarrow u(x) \sim u(y) \Leftrightarrow x \sim y$).
- CLF3 follows from $\mathrm{im}(W \circ u) \subseteq \mathrm{im}\, W$. □

Relation between Lyapunov functions and the coreflection of recurrent points

Complete Lyapunov functions recognize chain-recurrent points in the sense that if L is a complete Lyapunov function, there exists an equalizer diagram

$$Rec(X, f) \longrightarrow X \underset{L}{\overset{L \circ f}{\rightrightarrows}} \mathbb{R}$$

(or, in more mundane terms, $x \in Rec(X, f)$ if and only if $L(f(x)) = L(x)$ in $\mathbb{R}$.)

Remark

Immediate corollary of this, a space is full-recurrent if and only if for a fixed, and hence for any, complete Lyapunov function L, any map $A \to X$ equalizes (Lf, L).

Conley's Theorem, Categorically

Conley's Fundamental Theorem

Every compact dynamical system (X, f) admits a complete Lyapunov function; i.e. each fibre $\mathbf{CLF}_{(X,f)}$ is *non-empty*.

In categorical terms, Conley's theorem says:

- The fibration $P : \mathbf{CLF} \to \mathbf{Met}^{\mathbb{N}}$ has **non-empty fibres**.
- There is a *section* at the level of objects: for each (X, f) one can choose a complete Lyapunov function V.

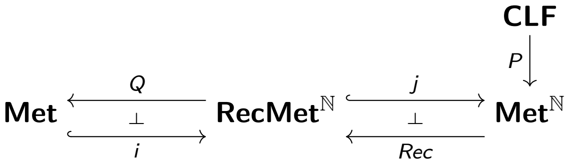
Conley's Theorem, Categorically

- The question of whether this section is *functorial* amounts to: can we choose $V_{(X,f)}$ consistently so that

$$V_Y(h(x)) \leq V_X(x) \quad \text{for every morphism } h : (X, f) \to (Y, g)?$$

- More stringent requirement: does each fibre $\mathbf{CLF}_{(X,f)}$ have a *terminal object*, and do reindexing maps preserve it?

Summary Diagram



- $P : \mathbf{CLF} \to \mathbf{Met}^{\mathbb{N}}$ is a posetal **OInj**-fibration (*today's main result*).
- $i \dashv R$: $\mathbf{RecMet}^{\mathbb{N}}$ is a **coreflective** subcategory of $\mathbf{Met}^{\mathbb{N}}$ (coreflector = restriction to chain-recurrent set).
- $L \dashv j$: $\mathbf{Met}$ is a **reflective** subcategory of $\mathbf{RecMet}^{\mathbb{N}}$ (reflector = quotient by chain-equivalence).
- $CQ = Q \circ Rec$ is **not** a left adjoint.

A 2-category from emergent order

Emergent Order Spectrum: Definition

Fix a compact topological dynamical system (X, f) with chain-related points x, y ($x \,\mathcal{C}\, y$).

- Build a sequence of **nested** and **acyclic** ϵ_n-chains $\{C_n\}$ from x to y, where $\epsilon_n \searrow 0$:

$$(C_0 \subseteq C_1 \subseteq C_2 \subseteq \dots) + (\epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \dots)$$

- For **order-compatible** sequences, the indices along the chains induce a linear order on the union of supports $S = \bigcup_n supp(C_n)$.

Definition

The EOS, denoted $\Omega(x, y)$, is the set of all **countable linear order-types** β obtained as direct limits of these sequences.

Emergent Order Spectrum

The Emergent Order Spectrum is a **robust, canonical invariant** characterized by:

- **Sequence Independence**: EOS does not depend on the specific sequence of errors $\{\epsilon_n\}$ chosen to construct the chains.
- **Metric Independence**: if (X, d) and (X, d') are equivalent metric spaces, $\Omega_{(X,d)}(x, y) \cong \Omega_{(X,d')}(x, y)$; EOS doesn't see equivalent metrics on X. More generally,
- **Conjugacy Invariance**: If h is a topological conjugacy between f and g, then $\Omega_f(x, y) = \Omega_g(h(x), h(y))$.
- **Emptiness when not recurrence**: $\Omega(x, y) = \emptyset$ if and only if x is not chain-related to y.

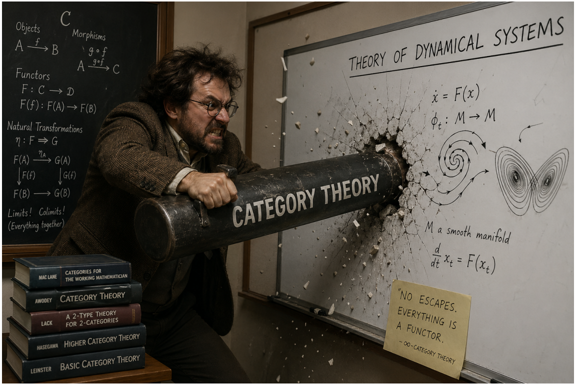
Emergent Order Spectrum

- **Orbit Detection**: A finite ordinal k is in $\Omega(x, y)$ if and only if $f^{k+1}(x) = y$.
- **Fixed Point Case**: If $f(x) = y$, then $\Omega(x, y) = \{\emptyset\}$ (the empty order-type).
- **Periodicity Detection**: A point x is periodic if and only if there exist finite ordinals $k \in \Omega(x, y)$ and $k' \in \Omega(y, x)$.
- **Order Complexity**: For transitive homeomorphisms[1] on a compact space X, the spectrum $\Omega(X^2) := \bigcup_{x,y \in X} \Omega(x, y)$ contains all countable **scattered orderings** and the countable order type of $\mathbb{Q}$.

[1] A homeomorphism $f : X \to X$ is **transitive** if for all $U, V \subseteq X$ there is n such that $f^n(U) \cap V$ is nonempty.

Intuition on $\Omega(x, y)$

This definition begs to be streamlined. . .



Intervals and the Category ∇

Definition

An **interval** is a finite totally ordered set with a distinct top $\top$ and bottom $\perp$; we write

$$\langle n\rangle = \{\perp < 1 < \cdots < n < \top\}, \quad n \geq 0$$

(so $\langle 0\rangle = \{\perp, \top\}$). The category ∇ has intervals as objects; morphisms $f : \langle n\rangle \to \langle m\rangle$ are *monotone maps preserving* $\top$ *and* $\perp$.

Theorem (Joyal)

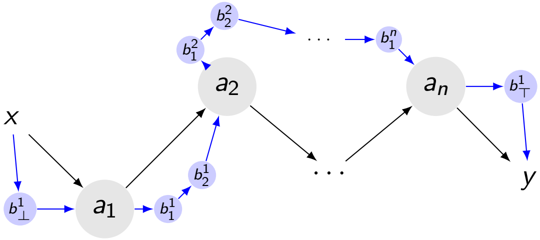
There is an equivalence of categories $D : \Delta^{op} \xrightarrow{\sim} \nabla$, acting on objects by $[n] \mapsto \Delta([n], [1])$.

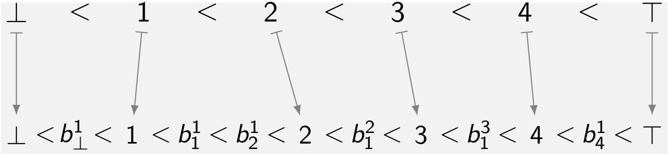
Acyclic ε-chains

Definition

An **acyclic ε-chain** from x to y in (X, φ) is an *injective* function $C : \langle n_\varepsilon \rangle \to X$ with $C(\bot) = x$, $C(\top) = y$, and the usual ε-step condition.

A **refinement** of an ε-chain:





The Hom-Categories $\mathbf{Met}^{🔗}_{(X,\varphi)}(x, x')$

Definition

For a dynamical system (X, φ) and points $x, x' \in X$, define the category $\mathbf{Met}^{🔗}_{(X,\varphi)}(x, x')$:

- *Objects*: pairs (ε, C) where $\varepsilon \geq 0$ and $C : \langle n_\varepsilon \rangle \to X$ is an acyclic ε-chain from x to x'.
- *Morphisms* $(\varepsilon, C) \to (\delta, C')$: pairs (p, f) where $p : \varepsilon \geq \delta$ is an inequality and $f : \langle n_\varepsilon \rangle \hookrightarrow \langle n_\delta \rangle$ is a *monomorphism* in ∇ (monotone, injective, preserving $\top$ and $\bot$) such that $C' \circ f = C$.

Unwinding: a morphism refines a coarser ε-chain C into a finer δ-chain C' (with $\delta \leq \varepsilon$) by embedding the support of C order-compatibly into the support of C'.

Two Forgetful Functors

The category $\mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x,x')$ carries two canonical forgetful functors:

1. *The precision functor*

$$e : \mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x,x') \longrightarrow \mathbf{Cost} := ([0,\infty],\geq)$$

sending $(\varepsilon, C) \mapsto \varepsilon$ and $(p,f) \mapsto p$. This makes $\mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x,x')$ a **fibration over Cost**.

2. *The length functor*

$$\Lambda : \mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x,x') \longrightarrow \mathbf{Lin}_{\aleph_0}$$

sending $(\varepsilon, \langle n_\varepsilon\rangle, C) \mapsto \langle n_\varepsilon\rangle$, where $\mathbf{Lin}_{\aleph_0}$ is the category of countable linear orders.

The Assembly $\int(\mathbf{Met}^{🔗})$

Applying the *Grothendieck construction* to $(X,\varphi) \mapsto \mathbf{Met}^{🔗}_{(X,\varphi)}$:

Definition (The category $\int(\mathbf{Met}^{🔗})$)

- *Objects*: tuples $((X; x, x', \varphi), (\varepsilon, C))$ where $(X; x, x')$ is a *bipointed* compact metric space, φ a dynamics, and (ε, C) an acyclic ε-chain from x to x'.
- *Morphisms* to $((Y; y, y', \psi), (\delta, C'))$: pairs $(h, (\geq, f))$ where $h : X \to Y$ is equivariant and non-expansive with $h(\{x, x'\}) \subseteq \{y, y'\}$, and $(\geq, f) : (\varepsilon, h_! C) \to (\delta, C')$ in $\mathbf{Met}^{🔗}_{hx,hx'}(Y, \psi)$ (i.e. $\varepsilon \geq \delta$ and f refines the pushed-forward chain).

The canonical functor $p : \int(\mathbf{Met}^{🔗}) \to \mathbf{Met}^{\mathbb{N}}_{\bullet\bullet}$ is an **opfibration**, with fibre over $(X; x, x', \varphi)$ equal to $\mathbf{Met}^{🔗}_{(X,\varphi)}(x, x')$.

Chains as a Diagram, and the Emergent Order

Fix $x, x' \in X$. A *refinement sequence* is a diagram

$$(\varepsilon_0, C_0) \xrightarrow{(\geq, f_0)} (\varepsilon_1, C_1) \xrightarrow{(\geq, f_1)} (\varepsilon_2, C_2) \xrightarrow{(\geq, f_2)} \cdots$$

in $\mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x, x')$, with $\varepsilon_n \searrow 0$, corresponding to a functor $S^{xx'}_{(X,\varphi)} : (\omega, \leq) \to \mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x, x')$.

Definition

The **emergent order** $\Omega(x, x')$ is (the image of) the colimit of the composite

$$(\omega, \leq) \xrightarrow{S^{xx'}_{(X,\varphi)}} \mathbf{Met}^{\text{🔗}}_{(X,\varphi)}(x, x') \xrightarrow{\Lambda} \mathbf{Lin}_{\aleph_0},$$

i.e. the linear order $C_\infty = \bigcup_{n \geq 0} C_n$ obtained by taking the colimit of the nested chain supports.

Conclusion

Conjectured 2-Categorical Structure

Conjecture (Conjecture A)

If (X, φ) is *full-recurrent*, then $\mathbf{Met}^{🔗}_{(X,\varphi)}$ assembles into a **2-semicategory** (only fixed points have identities) with:

- *Objects*: points $x \in X$;
- *1-cells* $x \to x'$: objects of $\mathbf{Met}^{🔗}_{(X,\varphi)}(x, x')$, i.e. acyclic ε-chains;
- *2-cells*: morphisms (p, f) in those hom-categories;
- *Composition* of 1-cells: concatenation/juxtaposition of chains.

Conjecture (Conjecture B)

The assignment $(X, \varphi) \mapsto \mathbf{Met}^{🔗}_{(X,\varphi)}$ extends to a functor

$$\mathbf{RecMet}^{\mathbb{N}} \longrightarrow \mathbf{2Cat}.$$