a hint of Chu

December 4, 2020

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between Boolean algebras and Stone spaces (totally disconnected, compact, T2);

- The Boolean algebra corresponding to a Stone space consists of its clopen sets.
- The Stone space associated to a Boolean algebra B is the set of its ultrafilters, equipped with a topology having as basis

$$\Big\{ V_b = \{ S \in \mathcal{F}(B) \mid b \in S \} \mid b \in B \Big\}.$$

This generates a 'Zariski' topology on the set of ultrafilters.

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Homming a topological space into $\underline{2}_S$ yields a functor **CHaus**$^{\mathrm{op}} \to$ **BA** that restricted to Stone spaces is an equivalence;

homming a Boolean algebra into $\underline{2}_B$ yields the set of ultrafilters, and also the Zariski topology can be recovered from **BA**$(E, \underline{2}_B)$.

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It is a set D that carries the structure of an $\mathcal{A}$-object and a $\mathcal{B}$-object, turning $\mathcal{A}, \mathcal{B}$ into equivalent categories via

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Principle
Interesting dualities arise from dualising objects. (Name your favourite one in your head, now)

Question: is there a general categorical framework in which dualities can be, if not subsumed, understood as parts of a general theory?

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Answer: yes.

The Chu construction

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To fix ideas, we shall concentrate on the case where $\mathcal{A}$ is the category of sets, and D a generic set. (More than often, and surely for all concrete models, $D = \{0, 1\}$)

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- Objects, the triples $\langle A, X, r : A \times Y \to D \rangle$ of sets A, X and a pairing function r; such a triple is a Chu space $\mathcal{S} = (A, X, r)$.

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- $\underline{\text{Chu}}(\mathbf{Set}, D)$ has a monoidal structure (complicated to describe)

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- there is a monoidal functor

$$i : \mathbf{Set} \hookrightarrow \underline{\text{Chu}}(\mathbf{Set}, D)$$

sending a set A into $A, X = D^A$, and $X \times D^A \to D$ is just evaluation.[1]

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$$X \xrightarrow{\eta} (X \times A)^A \xrightarrow{r^A} D^A$$

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	x	y	$x \wedge y$
a	1	0	0
b	0	1	0
$a \vee b$	1	1	?

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- A homomorphism of Chu spaces $(A, X, r), (B, Y, s)$ if the associated function

$$\hat{f} : 2^{D^A} \to 2^{D^B} : U \mapsto \{g : B \to D \mid gf \in U\}$$

is such that $\hat{f}(im\,\bar{r}) \supseteq im\,\bar{s}$ where

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Theorem
$f : A \to B$ is a homomorphism if and only if it is continuous.

A roundup of concrete examples

Sets

A Set is a normal Chu space image of the embedding

$$\textbf{Set} \to \underline{\text{Chu}}(\textbf{Set}, \underline{2})$$

So, a set is represented as a $|A| \times |2^A|$-matrix of 0's and 1's, one column for each subset of A.

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This is just a verbose way to bookkeep the powerset of A in a table whose columns are the characteristic functions $\chi_U : A \to 2$ of subsets of A.

Pointed sets

A pointed set is a set A with a distinguished element $a \in A$; given a Chu space (A, X, r) we represent the pointed Chu space $\mathcal{S}_+$ as $\mathcal{S}$ where we added a new row, constant at 0:

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Equivalently (!) we can pick an element $\bar{a} \in A$ and remove from $\mathcal{S}$ all the columns $E \in |2^A|$ for which $r(a, E) \neq 0$.

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- a normal Chu space $\mathcal{S}(A, X \subseteq \underline{2}^A, \underline{2})$ realizes a preorder if and only if the set of its columns is closed under arbitrary pointwise joins and meets;
- the property of being a partial order is a property of separation: $a \leq b$ and $b \leq a$ implies $a = b$.

Frames (of opens)

A topological space is an extensional Chu space whose columns are closed under arbitrary union and finite (including empty) intersection.

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A topological space is an extensional Chu space whose columns are closed under arbitrary union and finite (including empty) intersection. The Chu homomorphisms between topological spaces are exactly the continuous functions: whence the name continuous for a homomorpism $f : A \to B$.

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It works in other examples too.

It does not work always: embedding (abelian) groups asks for ‘big’ representation alphabet.

In fact, there is no embedding of the category of groups in $\underline{\text{Chu}}(\mathbf{Set}, D)$ for any *finite* set D; (what about an infinite set?)

There are nice embedding results of categories relevant to topology and Quantum Mechanics into $\underline{\text{Chu}}(\mathbf{Set}, I)$ where I is a closed interval of $\mathbb{R}$.

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We can read the above positive and negative results 'backwards': there is always a faithful functor

$$j : \mathcal{C} \to \underline{\mathrm{Chu}}(\mathbf{Set}, 2) : C \mapsto \langle UC, UC \times \mathcal{C}(C, 2) \to 2 \rangle$$

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a sufficient condition for j to be also full is that the pair $UC \times \mathcal{C}(C, 2)$ 'completely determines' the $\mathcal{C}$-structure on C.

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the carrier $|G|$ and the set of group homomorphisms $\hom(G, \underline{2})$ does not determine the whole structure of G;

for example, it is impossible to decide whether $G = C_{25}$ or $G = C_5 \times C_5$ from the fact that $\hom(G, C_2) = 0$ and $|G| = 25$

Free space for discussion

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