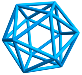
On the unicity of the formal theory of categories

Fosco Loregian



Max Planck Institute for Mathematics

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- Every comment or advice is welcome.

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  - Two-sided Yoneda structure
  - Isbell duality and its generalizations

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  - Representable Yoneda structure
  - Two-sided Yoneda structure
  - Isbell duality and its generalizations
- An open and motivating problem: derivators.

Introduction

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«The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well as to other parts of mathematics.»

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2.

«The basic idea is that the category of small categories, **Cat**, is a "2-category with properties"; one should attempt to identify those properties that enable one to do the "structural parts of category theory".»

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- cf. topos theory: treat $\mathcal{E}$ as if it were **Set**;
- cf. categorical algebra: treat $\mathcal{A}$ as if it were $\mathsf{Alg}(\mathbb{T})$;
- cf. homological algebra: treat $\mathcal{C}$ as if it were $\mathsf{Ch}(\mathcal{A})$;

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Sure you can do many things:

- adjunctions: pairs of 1-cells $f : X \leftrightarrows Y : g$
- monads: endo-1-cells $t : A \to A$
- Kan extensions
- fibrations

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- a "pointwise" formula to compute Kan extensions
- equivalent characterization of adjunctions: $Y(f,1) \cong X(1,g)$

Y) the Yoneda lemma ("category theory is the Yoneda lemma")

E) a calculus of modules ("category theory is the theory of multi-object monoids")

Idea

Take a 2-category $\mathcal{K}$, and take 1. and 2. very seriously.

(Y): Yoneda structures

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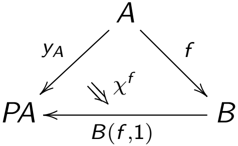
- A *Yoneda structure* imposes on $\mathcal{K}$ enough structure so that every "small" object A has a "Yoneda embedding" $y_A : A \to PA$.
- Formally encode the Yoneda lemma in universal properties of certain diagrams



representing f-nerves.

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- This leads to the notion of C-D-bimodule as a functor $C^\vee \times D \to \mathit{Set}$. All such bimodules live in a bicategory Mod, and there is an embedding

$$p : \mathit{Cat} \hookrightarrow \mathit{Mod}$$

such that every $f \in \mathit{Cat}$ has a right adjoint when embedded in Mod.

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- This leads to the notion of C-D-bimodule as a functor $C^\vee \times D \to Set$. All such bimodules live in a bicategory Mod, and there is an embedding

$$p : Cat \hookrightarrow Mod$$

  such that every $f \in Cat$ has a right adjoint when embedded in Mod.
- Every such $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows.

The main theorem of this talk is an equivalence between a certain class of Yoneda structures and a certain class of proarrow equipments:

$$\left\{ \begin{matrix} \text{cocomplete} \\ \text{Yoneda} \\ \text{structures} \end{matrix} \right\} \Longleftarrow\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= \left\{ \begin{matrix} \text{Yoneda} \\ \text{equipments} \end{matrix} \right\}$$

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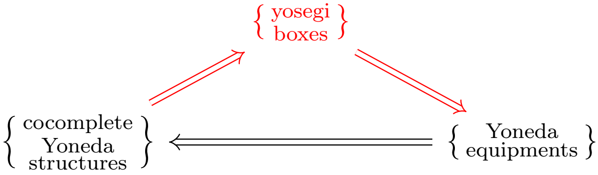

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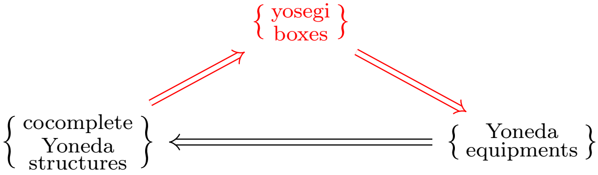
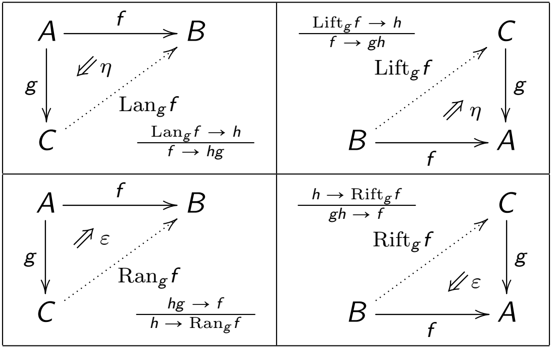


What allows to pass from YS to E is the notion of a yosegi, a certain special kind of 2-monad.

Preliminaries

2-dimensional universality

The main 2-dimensional universal constructions we are interested in



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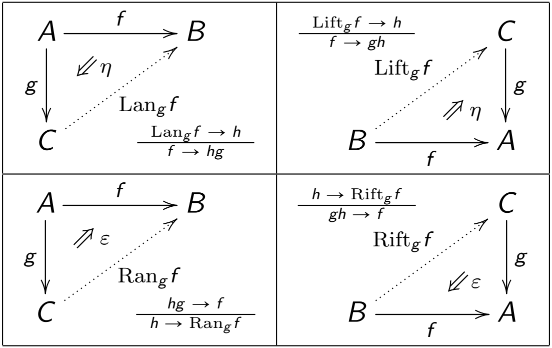

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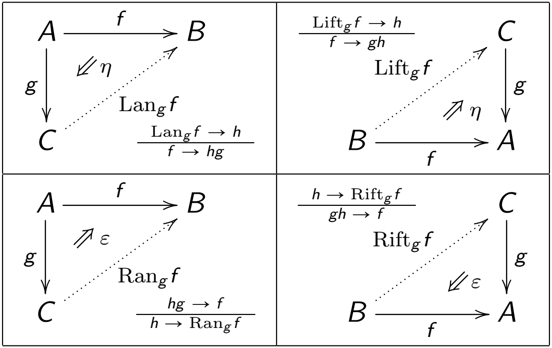

pointwise. . . . . . absolute

Proposition (formal description of adjoints)

The following conditions are equivalent, for a pair of 1-cells $f : A \leftrightarrows B : g$:

- $f \dashv g$ with unit η and counit ϵ;
- The pair $\langle g, \eta \rangle$ exhibits the absolute Lan of 1 along f
- The pair $\langle g, \eta \rangle$ ehhibits the Lan of 1 along f, and f preserves it.

A bit of coend calculus:

Proposition (ninja Yoneda lemma)

For every presheaf $F : \mathcal{C} \to \mathbf{Set}$, it holds

$$F(X) \cong \int_A FX^{\hom(A,X)} \qquad F(X) \cong \int^A FA \times \hom(X, A)$$

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Proposition (pointwise Kan extensions)

For a diagram $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$, it holds

$$\mathrm{Lan}_G F(X) \cong \int^A \hom(GA, X) \times FA$$

naturally in X.

Yoneda structures

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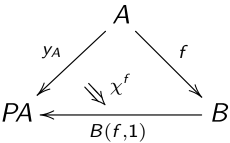
  - A family of triangles



filled by a 2-cell $\chi : y_A \to B(f, f) =: B(f, 1) \cdot f$.

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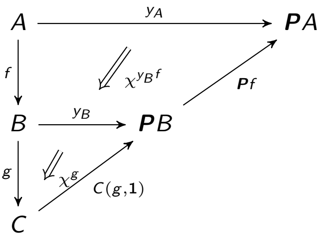
  - Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells



exhibits the pointwise extension $\mathrm{Lan}_{gf} y_A = C(gf,1)$ (shortly, $\boldsymbol{P}$ is a functor).

In **Cat**

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- y_A is of course the Yoneda embedding $A \to [A^{\mathrm{op}}, \mathbf{Set}]$;
- χ^f is obtained from the action of f on arrows.

In **Cat**

Axiom 1

$B(f, 1) = \lambda b.\lambda a.\,\hom_B(fa, b)$ exhibits with χ^f the pointwise left Kan extension of y_A along $f : A \to B$.

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It is enough to check that

$$\begin{aligned}
[B, PA](N_f, G) &\cong \int_b PA(B(f,b), Gb) \\
&\cong \int_{ab} \mathbf{Set}(B(fa,b), G(b)(a)) \\
&\cong G(fa)(a) \\
[A, PA](y_A, G \cdot f) &\cong \int_a PA(y_A(a), G(fa)) \\
&\cong G(fa)(a).
\end{aligned}$$

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&\cong \int_{a'} [A^{\text{op}}, \mathbf{Set}]\big(y_A a', B(f-, ga')\big) \\
&\cong \int_{a'} B(fa', ga') \\
&\cong [A, B](f, g)
\end{aligned}$$

absoluteness can be checkd by hand.

In **Cat**

Axiom 3-4

They admit a rephrasing as $\boldsymbol{P}$ is a functor, in that $\boldsymbol{P}(id) \cong id$ and $\boldsymbol{P}(gf) \cong \boldsymbol{P}f \cdot \boldsymbol{P}g$.

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This is pretty obvious (although the isomorphisms

3) $[\boldsymbol{P}A, \boldsymbol{P}A](1, H) \cong [A, \boldsymbol{P}A](y_A, H \cdot y_A)$;
4) $C(gf, 1) = \mathrm{Lan}_{gf} y_A \cong \mathrm{Lan}_{y_B f} y_A \circ \mathrm{Lan}_g y_B = \boldsymbol{P}B(y_B f, 1) \cdot C(g, 1)$

can be checked directly).

All **Cat**-like 2-categories carry Yoneda structures in the obvious way:

- $\mathcal{V}$-**Cat** (enriched categories) with the enriched Yoneda embeddings;
- internal categories in a finitely complete category $\mathcal{A}$;
- the 2-category of pseudofunctors $\mathcal{A} \to$ **Cat** for a small bicategory $\mathcal{A}$;
- ...

Proarrow equipments

Definition (proarrow equipment)

A 2-functor $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows if

- p is the identity on objects and locally fully faithful;
- p is such that for each 1-cell $f : A \to B$ in $\mathcal{K}$ $p(f)$ has a right adjoint in $\overline{\mathcal{K}}$.

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Very simple definition; a bit complicated to follow the literature. Recently framed into the more natural environment of (hyper)virtual double categories of (Shulman-Crutwell).

Question

How do these framework relate? It is reasonable to expect they do (they're both ways to encode a calculus of profunctors; $y_A : A \to \boldsymbol{P}A$ is the (mate of the)identity profunctor).

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3. if only it was possible to go back to a Yoneda structure...

Turns out 1. is almost true; 2. is true; 3. is true if we restrict to so-called Yoneda equipments where p has additional properties.

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3. the monad $\boldsymbol{P}_!$ is a KZ-doctrine.

Definition (yosegi box)

Let $j : \mathcal{A} \subset \mathcal{K}$ and $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with the same properties, and such that moreover

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We call $\boldsymbol{P}$ a (j-relative) yosegi box.[a]

[a] *Yosegi-zaiku* (寄木細工) is a kind of marquetry featuring elaborate inlaid and mosaic designs; the defining properties of such a KZ-doctrine are tightly linked, rich of peculiar adornments.

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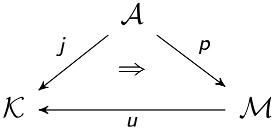
2. the inclusion $j : \mathcal{A} \subset \mathcal{K}$ admits a *Yoneda equipment* $p\ {}_j\!\dashv u$;
3. The inclusion $j : \mathcal{A} \subset \mathcal{K}$ *extends to a yosegi box*.

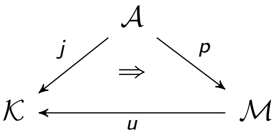
- a Yoneda equipment is a triangle



such that

  - p is a proarrow equipment à la Wood;
  - p is the j-relative left adjoint of $u : \mathcal{M} \to \mathcal{K}$;
  - the relative monad up generated by the relative adjunction, is lax idempotent with unit $\eta : j \Rightarrow up$.

- a Yoneda equipment is a triangle



such that
  - p is a proarrow equipment à la Wood;
  - p is the j-relative left adjoint of $u : \mathcal{M} \to \mathcal{K}$;
  - the relative monad up generated by the relative adjunction, is lax idempotent with unit $\boldsymbol{\eta} : j \Rightarrow up$.
- the inclusion j extends to a yosegi if there is a relative lax-idempotent 2-monad $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with relative unit $j \Rightarrow \boldsymbol{P}$ such that for each 1-cell $f : A \to B$, the 1-cell $\boldsymbol{P}(f)$ has a left adjoint $\boldsymbol{P}_! f$.

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- every cocomplete Yoneda structure has a yosegi as presheaf construction;
- Yoneda equipments contain enough information to recover a Yoneda structure.

Applications

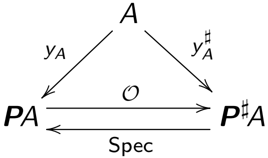
Examples

- Often, $\boldsymbol{P}$ is representable, because $\mathcal{K}$ is cartesian closed: $\boldsymbol{P} : A \mapsto [A^{\mathrm{op}}, \Omega]$; this is the case for many $\mathcal{K}$'s having a duality involution $(\_)^{\mathrm{op}}$, and entails that $\boldsymbol{P}$ has an adjoint

$$\boldsymbol{P}^\sharp \dashv \boldsymbol{P}$$

In **Cat**, $\boldsymbol{P}^\sharp$ is the "contravariant presheaf construction", sending A to $[A, \mathbf{Set}]^{\mathrm{op}}$.

- This self-duality of $\mathbf{P}$ is tightly linked to the possibility to instantiate the formal version of *Isbell duality*: there is an adjunction



where $\mathcal{O} = \mathrm{Lan}_{y_A}(y_A^\sharp)$ and Spec $= \mathrm{Lan}_{y_A^\sharp}(y_A)$.

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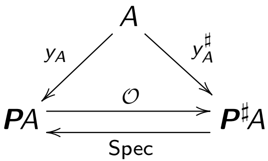

  where $\mathcal{O} = \mathrm{Lan}_{y_A}(y_A^\sharp)$ and $\mathrm{Spec} = \mathrm{Lan}_{y_A^\sharp}(y_A)$.

- The pair $(\boldsymbol{P}^\sharp, \boldsymbol{P})$ forms a two-sided Yoneda structure/yosegi: we have copresheaf constructions working as free completion:
  $\boldsymbol{P}^\sharp : A \mapsto [A, \mathbf{Set}]^{\mathrm{op}}$ does precisely this in **Cat**; the axioms of a (left) Yoneda structure hold replacing left extensions and lifts with the corresponding right versions. These two structures are compatible.

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$$\begin{array}{ccccc} \mathbf{cat} & \xrightarrow{\nu} & \mathbf{Cat}_\infty & \longrightarrow & \mathrm{Der} \\ {\scriptstyle y}\downarrow & & & \nearrow & \\ [\mathbf{cat}^{\mathrm{op}}, \mathbf{cat}] & & & & \end{array} \qquad \begin{array}{ccccc} A & \longmapsto & \mathbf{sSet}^{NA} & \longmapsto & \mathrm{Ho}(\mathbf{sSet}^{NA}) \\ \downarrow & & & & \\ \boldsymbol{y}(A) & & & & \end{array}$$

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The arrow y is the Yoneda embedding of small categories into small prederivators; taking the Yoneda extension of the functor $\nu : A \mapsto \mathrm{sSet}^{NA}$ now we get a functor from pDer (lowercase p = small prederivators) to (presentable) ∞-categories, whose homotopy categories are thus derivators.