Functorial semantics for partial theories

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Seminari di Logica Torino–Udine

Introduction

joint work with

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A category-theorist' view on universal algebra.

> Opinions are my own.
> — the Twitter angry mob

What's the paper about? A variety theorem for partial algebraic theories (operations are partially defined).

Plan of the talk

1a. A semiclassical look to algebraic theories;

1b. Essentially algebraic theories;

2a. Towards a Lawvere-like characterization;

2b. the variety theorem of PLTs.

Two take-home messages

The classical picture

You could have invented universal algebra
if only you knew category theory

if you define it right, you won't need a subscript.
— Sammy Eilenberg

Theories

Fact: the category $\mathbf{Fin}^{\mathrm{op}}$ is the free completion of $\bullet$ under finite products.

$$[n] \in \mathbf{Fin} \qquad [n] = [1] + \cdots + [1] \quad n \text{ times}$$

Definition (Lawvere theory)
A Lawvere theory is a functor $p : \mathbf{Fin}^{\mathrm{op}} \to \mathcal{L}$ that is identity on objects (='idonob') and strictly preserves products.

$$p(n+m) = p[n] \times p[m]$$

There is a category **Law** of Lawvere theories and morphisms thereof.

Theories

Theorem

It is equivalent to give

1. *a Lawvere theory, in the sense above. (Lawvere, 1963)*
2. *a finitary monad T on* **Set**. *(Linton, 1966)*
3. *a finitary monadic LFP category over* **Set** *(Adàmek, Lawvere, Rosickỳ 2003)*
4. *a cartesian operad $P : \mathbf{Fin} \to \mathbf{Set}$ – a certain monoid $\left(\substack{P \diamond P \to P \\ 1 \to P} \right)$ in $[\mathbf{Fin}, \mathbf{Set}]$ (probably known to Lawvere?).*
5. *a cocontinuous monad T on $[\mathbf{Fin}, \mathbf{Set}]$, which is convolution monoidal. (cmc := convolution monoidally cocontinuous)*

$1 \iff 2 \iff 3$

Let $p : \mathbf{Fin}^{\mathrm{op}} \to \mathcal{L}$ be the theory, and consider the strict pullback in Cat:

$$\begin{array}{ccc} M(\mathcal{L}) & \longrightarrow & [\mathcal{L}, \mathbf{Set}] \\ {\scriptstyle U}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle -\circ p} \\ \underset{A}{\mathbf{Set}} & \longrightarrow & \underset{\lambda n.A^n}{[\mathbf{Fin}^{\mathrm{op}}, \mathbf{Set}]} \end{array}$$

it's a pullback in the 2-category of locally fin. presentable categories & left adjoints; $M(\mathcal{L})$ is thus locally fin. presentable, and a left adjoint $F \dashv U$ can be built by hand.

In addition, $- \circ p$ is monadic $\Rightarrow$ U is monadic, thus $M(\mathcal{L}) \cong Alg(UF)$ is monadic. Finally, U is finitary $\Rightarrow$ UF is finitary.

$$M(\mathcal{L}) = \text{models of the theory } (p, \mathcal{L})$$

$1 \iff 2 \iff 3$

Given a finitary monad T on **Set**, consider the composition

$$\mathbf{Fin} \longrightarrow \mathbf{Set} \longrightarrow Kl(T)$$

(free T-algebras on finite sets). Factor this functor as a composition of an idonob + a fully faithful functor, and discard the fully faithful one.

$$\begin{array}{ccc} \mathbf{Fin} & \xrightarrow{J} & \mathbf{Set} \\ {\scriptstyle p}\downarrow {\scriptstyle bo} & & \downarrow {\scriptstyle F^T} \\ \mathcal{L} & \xrightarrow[ff]{} & Kl(T) \end{array}$$

This gives a Lawvere theory, whose algebras (by a theorem of Linton) are exactly the T-algebras.

$2 \iff 4$

Any monad $T : \mathbf{Set} \to \mathbf{Set}$ determines a functor $T \circ J : \mathbf{Fin} \to \mathbf{Set}$ by restriction along $J : \mathbf{Fin} \to \mathbf{Set}$; if (and only if) T is finitary, it can be recovered from $T \circ J$ via left extension:

$$\mathrm{Lan}_J(TJ) \cong T$$

This is encapsulated in an equivalence of categories

$$[\mathbf{Set}, \mathbf{Set}]_\omega \underset{\mathrm{Lan}_J}{\overset{- \circ J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathbf{Fin}, \mathbf{Set}]$$

which can be promoted to a monoidal equivalence $([\mathbf{Set}, \mathbf{Set}]_\omega, \circ) \cong ([\mathbf{Fin}, \mathbf{Set}], \diamond)$ transporting $\circ$; a clone is exactly a $\diamond$-monoid, and a finitary monad is exactly a $\circ$-monoid in $[\mathbf{Set}, \mathbf{Set}]_\omega$.

$4 \iff 5$

($\Rightarrow$) In every monoidal category, tensoring with a monoid yields a monad. In this case, it is cmc.

($\Leftarrow$) Let $S : [\mathbf{Fin}, \mathbf{Set}] \to [\mathbf{Fin}, \mathbf{Set}]$ be a cmc monad; then it is determined by its action on representables and

$$\begin{aligned}
(SA)m &\cong S\left(\int^{n} An \times \mathbf{Fin}(n, m)\right) && \text{(Yoneda)} \\
&\cong \int^{n} An \times S\mathbf{Fin}(n, m) && \text{(cocont)} \\
&\cong \int^{n} An \times S(\mathbf{Fin}(1, m)^{*n}) && \text{(def. of } *) \\
&\cong \int^{n} An \times (S\mathbf{Fin}(1, m))^{*n} && \text{(}*\text{-preserving)} \\
&\cong \left(\int^{n} An \times (SJ)^{*n}\right) m \cong (A \diamond SJ)m
\end{aligned}$$

so that S is uniquely determined as $- \diamond SJ$.

Distributive laws btwn theories

This perspective can be pushed quite far: [Cheng] adapting a former result of Rosebrugh and Wood

Theorem
Let $\mathcal{C}$ be a small category; a factorisation system $(\mathcal{E}, \mathcal{M})$ on it is precisely a pair of monads $\mathcal{E}$ and $\mathcal{M}$ in **Span** *together with a distributive law of $\mathcal{E}$ over $\mathcal{M}$ such that the composite monad $\mathcal{M} \cdot \mathcal{E}$ is the category $\mathcal{C}$ (Memento: categories are monads in Span).*

[Cheng] : distributive laws between Lawvere theories correspond to factorisation systems 'modulo $\mathbf{Fin}^{\mathrm{op}}$', in such a way that

- a distributive law for a Lawvere theory
- a distributive law for the associated finitary monad

correspond bijectively.

Power enriched theories

Following [Pow99] a $\mathcal{V}$-enriched theory is a idonob $\mathcal{V}$-functor $p : \mathcal{V}_{<\omega} \to \mathcal{L}$ where $\mathcal{V}_{<\omega}$ is the category of finitely presentable objects of an LFP, monoidal closed base of enrichment.

An algebra (or model) for a theory p is a finite cotensor-preserving functor $\mathcal{L} \to \mathcal{V}$; this defines the $\mathcal{V}$-category of models of p.

Many characterizations transport untouched: a $\mathcal{V}$-enriched theory is also

- a cmc monad $T : [\mathcal{V}_{<\omega}, \mathcal{V}] \to [\mathcal{V}_{<\omega}, \mathcal{V}]$;
- a finitary $\mathcal{V}$-monad on $\mathcal{V}$;
- a finitary monadic-$\mathcal{V}$-LFP category of $\mathcal{V}$.

Theories as $\mathcal{W}$-categories

Recent work of Garner [BG] builds on the equivalence between finitary endofunctors of **Set** and $[\mathbf{Fin}, \mathbf{Set}]$.

Taking the category $[\mathbf{Fin}, \mathbf{Set}]$ works as base of enrichment, and blurring the distinction between the categories $[\mathbf{Fin}, \mathbf{Set}] \cong [\mathbf{Set}, \mathbf{Set}]_\omega = \mathcal{W}$:

- A finitary monad is a monoid in $\mathcal{W}$, i.e. a $\mathcal{W}$-category with a single object;
- A Lawvere theory is a $\mathcal{W}$-category that is absolute (=Cauchy-, =Karoubi-)complete as an enriched category and generated by a single object.

Lawvere theories form a reflective subcategory in finitary monads; reflection is the enriched Cauchy completion functor.

Theories as $\mathcal{W}$-categories

In this perspective

there is no difference between a Lawvere theory and its associated monad: they are the very same thing, up to a Cauchy-completion operation.

Note also that:

- The Cauchy completion of a monoid in Cat is rarely a monoid: take the "generic idempotent" $M = \{1, e\}$ and split $e : * \to *$ as $r : 0 \leftrightarrows * : s$.
- In order to add all $\mathcal{W}$-absolute colimits, at least all tensors $y[n] \odot X$ must be added to the single object X.

Theories as $\mathcal{W}$-categories

Equivalently,

- A Lawvere $\mathcal{W}$-category is a $\mathcal{W}$-category which
  - is freely generated by cotensors with a single object X: $y[1] \odot X, y[2] \odot X, \ldots;$
  - admits all $\mathcal{W}$-absolute colimits.
- A $\mathcal{W}$-category is a special kind of cartesian multicategory: one where a multimorphism $f : X_1 \ldots X_n \to X$ is such that $X_1 = X_2 = \cdots = X_n = X$.

Recent work of Bourke and Garner builds a generalised monad-theory correspondence (by construction, this can't be made more general):

- A pretheory is an idonob functor $J : \mathcal{A} \to \mathcal{V}$; its codensity monad $Ran_J J$ is a monad on $\mathcal{V}$;
- given a monad on $\mathcal{V}$, consider the idonob part of $\mathcal{A} \to \mathcal{V} \to Kl(T)$.

The equivalence of categories induced by the fixpoints of this adjunction is 'the' monad-theory correspondence.

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (F \dashv G)$$

$$Fix(GF) \underset{\sim}{\rightleftarrows} Fix(FG)$$

Other sources

- Lucyshyn-Wright for a sharp characterization of ‘eleutheric theories’ as $\mathcal{J}$-ary monads or semi-representable profunctors;
- see Johnstone & Wraith for internal algebraic theories (e.g. in a topos);
- see Fuji, for an encompassing notion of ‘theory’ as ‘monoid in a place’, and for a notion of ‘meta’theory and ‘meta’model.

...definitely a longer story than this brief account can contain!

A timeline of functorial semantics (of algebraic theories)

1963 Lawvere's 'Functorial semantics'

1966 Linton theories are monads

1966 Linton: infinitary theories

1968 Ehresmann-Gabriel-Ulmer: sketches

1969 Linton: a category of arities

1971 (Gabriel-) Ulmer: LPcat

1978 Theories in a topos/as promonads

1999 Enriched theories

2009 "Theories" in double categories

2011 Distributive laws b'wn theories

2013 Theories are F-enriched cats

2015 Eleutheric theories

2018 Monads and (pre)theories

2019 Theories in fibered cats

Essentially algebraic theories

Definition (EAT)
An essentially algebraic theory is a quadruple

$$(\Sigma, E, \Sigma_t, D)$$

where the pair (Σ, E) is an equational theory, $\Sigma_t \subseteq \Sigma$ is a subset of 'totally defined' operation symbols, and D is a function on $\Sigma \setminus \Sigma_t$ accounting for the domain of definition of partially defined operations.

A model of an EAT is a model of the equational theory (Σ, E) with the property that every $\tau \in \Sigma_t$ is everywhere defined, and every $\sigma \in \Sigma \setminus \Sigma_t$ is interpreted as a partial function (with domain specified by $D(\sigma)$).

3.34 Definition

(1) An *essentially algebraic theory* is a quadruple

$$\Gamma = (\Sigma, E, \Sigma_t, \mathrm{Def})$$

consisting of a many-sorted signature Σ of algebras, a set E of Σ-equations, a set $\Sigma_t \subseteq \Sigma$ of "total" operation symbols, and a function Def assigning to each operation symbol $\sigma \colon \prod_{i \in I} s_i \to s$ in $\Sigma - \Sigma_t$ a set $\mathrm{Def}(\sigma)$ of Σ_t-equations in the standard variables $x_i \in V_{s_i}$ $(i \in I)$.

(2) We say that the theory Γ is *λ-ary*, for a regular cardinal λ, provided that Σ is λ-ary, each of the equations of E and $\mathrm{Def}(\sigma)$ uses less than λ standard variables, and each $\mathrm{Def}(\sigma)$ contains less than λ equations.

(3) By a *model* of an essentially algebraic theory Γ we mean a partial Σ-algebra A such that

(a) A satisfies all equations of E,

(b) for each $\sigma \in \Sigma_t$, the operation σ_A is everywhere defined,

(c) for each $\sigma \in \Sigma - \Sigma_t$ with $\sigma \colon \prod_{j \in J} s_j \to s$ and any $a_j \in A_{s_j}$ $(j \in J)$ we have that $\sigma_A(a_j)$ is defined iff A satisfies all equations of $\mathrm{Def}(\sigma)$ in the elements a_j.

The category of all models and homomorphisms is denoted by $\mathbf{Mod}\,\Gamma$. A category is called *essentially algebraic* if it is equivalent to $\mathbf{Mod}\,\Gamma$ for some essentially algebraic theory Γ.

[AR94]

Theorem (Gabriel-Ulmer duality)

There is a biequivalence of 2-categories

$$\mathbf{Lex}^{op} \leftrightarrows \mathbf{LFP}$$

between

- **Lex**, *the 2-category of small categories with finite limits, where 1-cells are functors preserving finite limits and 2-cells are the natural transformations, and*
- **LFP**, *the 2-category of locally finitely presentable categories, where 1-cells are right adjoints preserving directed colimits.*

A syntax-Semantics duality for EATs

GU duality prescribes the rule under which

- every EAT has an associated finite limit theory, whose category of models is LFP;
- Conversely, every LFP determines a category with finite limits, the opposite of FP objects, and this is an EAT.

- Syntax: a class of small categories defined by a sketch of shapes;
- Semantics: a class of large categories molded by syntax.

$$\textbf{Syntax} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \textbf{Semantics}$$

Mindful of Eilenberg's principle, we now wonder: is there an analogue of 1-6 above for essentially algebraic theories?

Is there an equational notion of theory, that we can use to build a syntax-semantics correspondence on the lines of 1-6?

Short answers: yes, we already have finite limit theories, but

> *the 'doctrine' of finitely complete categories does not, per se, provide a notion of syntax to replace classical terms, nor a calculus for (partial) equational reasoning about the categories of models they define.*

Short answer 2: we can fix this.

End of Part I

A failed approach

Fact: there is an equivalence of categories

$$\mathbf{Set}_* \cong \partial\mathbf{Set}(= \mathbf{Par})$$

between pointed sets are sets and partial functions.

Idea: Exploit this to translate questions about partial functions of sets into questions about pointed sets, i.e. into $\mathbf{Set}_*$-enriched functorial semantics in the sense of Power et al.

Problem: there appears to be no 'Linton theorem' linkin theories and monads.

In hindsight, we were working in the wrong 2-category.

Fortunately there is a setting that was engineered to axiomatise the features of pointed sets / partial functions.

A working approach

A restriction category is a category $\mathcal{C}$ with a restriction structure: a coherent choice of an idempotent $\bar{f} : A \to A$ for each morphism $f : A \to B$, satisfying certain axioms.

There is a 2-category **rCat** of restriction categories, restriction functors and transformations.

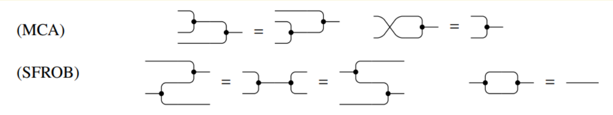
This 2-category **rCat** is a whole new world (a pretty ugly one, if you ask me). [Cocket and Lack 1, 2, 3,...]

It is however the only way (afa we know today) to provide a 'Lawvere-style' notion of essential algebraic theory.

Fortunately we can resort to yet another equivalent presentation for our partial Lawvere theories, that exploits as little machinery of restriction categories as it is possible.

- a partial Lawvere theory is the analogue of a cartesian functor in the restriction world
- every PLT has a category of models, a LFP category, but for us $\mathbf{Mod}(p)$ is regarded over **Par**.

A *partial Frobenius algebra* $(A, \delta_A, \mu_A, \varepsilon_A)$ in a symmetric monoidal category consists of a commutative comonoid $(A, \delta_A, \varepsilon_A)$ and a commutative semigroup (A, μ_A) interacting together as follows: the co/mult satisfy the equations



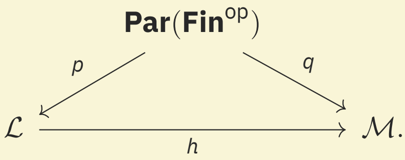
A discrete cartesian restriction category[1] is the same thing as a symmetric monoidal category where every object A is equipped with a coherent partial Frobenius algebra structure $(A, \delta_A, \varepsilon_A, \mu_A)$ with natural comultiplication.

[1]The kind of restriction categories we are interested into

Partial theories and their models

A partial Lawvere theory is a DCR category $\mathcal{L}$ for which there is an identity-on-objects DCR functor $\mathbf{Par}(\mathbf{Fin}^{\mathrm{op}}) \to \mathcal{L}$.

A morphism of partial Lawvere theories is a functor $h : \mathcal{L} \to \mathcal{M}$ s.t. the following triangle commutes:



This defines the category **pLaw** of partial Lawvere theories.

Partial theories and their models

Mimicking also the definition of model of a Lawvere theory, we obtain at once the notion of model of a partial Lawvere theory:

Definition (Model of a partial Lawvere theory)

A model for a partial Lawvere theory $\mathcal{L}$ is a CR functor $L : \mathcal{L} \to \mathbf{Par}$. A homomorphism $L \to L'$ is a lax natural transformation $\alpha : L \Rightarrow L'$.

DCRC to the rescue

The connection between our theorem and the finit-limit-theories approaches relies on Gabriel-Ulmer duality:

Proposition

- If $\mathbb{C}$ is a category with finite limits, $\mathsf{Par}(\mathbb{C})$ is a DCR category.

A converse holds:

- Every restriction category has a subcategory of total maps, and if $\mathcal{C}$ is Cauchy-complete then $\mathbb{C} = tot(\mathcal{C})$ has finite limits.

This sets up an adjunction (in fact, a reflection of **Lex**, categories with finite limits in **DCRC**).

The variety theorem

Theorem
There is a 2-adjunction

$$\mathbf{Th} : \mathbf{LFP} \leftrightarrows (\mathbf{DCRC}^{\leq})^{op} : \mathbf{Mod},$$

where $\mathbf{DCRC}^{\leq}$ *is the 2-category of DCR categories, CR functors and lax transformations and* **LFP** *is the 2-category of LFP categories, finitary right adjoints and nat. transformations.*

The unit of this adjunction is an equivalence, i.e.

$$\mathcal{K} \simeq \mathbf{Mod}(\mathbf{Th}(\mathcal{K}))$$

i.e. each LFP category is equivalent to the category of models of its induced theory.

Proof

Define an adjunction

$$K_t : \mathbf{DCRC}^{\leq} \leftrightarrows \mathbf{Lex} : \mathbf{Par}$$

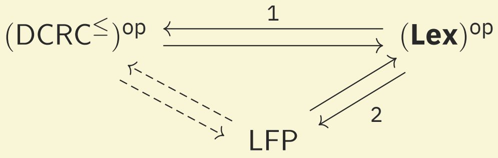
K_t acts as follows: given a DCRC $\mathbb{X}$, $K_t(\mathbb{X})$ has

- objects pairs (A, a) with A an object of $\mathbb{X}$ and $a : A \to A$ a domain idempotent in $\mathbb{X}$.
- arrows $f : (A, a) \to (B, b)$ are arrows $f : A \to B$ of $\mathbb{X}$ such that $\overline{f} = a$ and $bf = f$.
- composition is given by composition in $\mathbb{X}$.
- The identity on (A, a) is given by a.

$K_t(\mathbb{X})$ has finite limits (painstakingly compute binary products and equalisers by hand).

- The functor **Par** acts sending a category with finite limits into its category of partial maps: spans of morphisms in $\mathcal{C}$ where a leg is a monomorphism. This is a Cauchy-complete DCR category.
- The two functors arrange in an adjunction $K_t \dashv$ **Par**, and the counit of this adjunction is invertible, giving that **Lex** is a 2-reflective 2-subcategory of $\mathbf{DCRC}^{\leq}$.

This closes the diagram of adjunctions



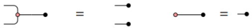
and proves the theorem.

Examples

Partial structures

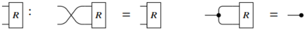
Example 5.4 ((Partial) Commutative Monoids). We start with the monoidal theory of commutative monoids (Example 2.16), where the multiplication and unit generators are re-coloured to red to avoid a clash. In models, the multiplication operation may be partially defined *and* the unit may be undefined. To define the partial theory of *total* commutative monoids, we'd need to add equations:

(3)

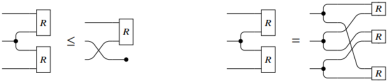


Equivalence relations

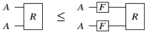
Example 5.6 (Equivalence Relations)**.** Consider the partial Lawvere theory consisting of a single binary operation R with coarity 0, together with equations expressing symmetry and reflexivity:



Note that *inequations* of terms, as in Remark 3.4, do not add expressivity. As such, we may use them freely when specifying partial Lawvere theories. Transitivity is intuitively captured by the inequation on the left, which, unfolding the definition of $\leq$, is precisely the equation on the right:



A model $\mathcal{A}$ of this theory consists of a set A together with an equivalence relation $=_A \subseteq A \times A$ corresponding to the domain of definition of $\mathcal{A}(R)$. A morphism $F : \mathcal{A} \to \mathcal{B}$ is a function $F : A \to B$ with $a =_A b \Rightarrow Fa =_B Fb$, which arises from the requirement that F is a lax transformation:

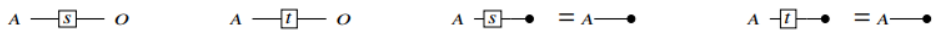


Thus, the variety corresponding to this theory is the category of *Bishop sets* (*setoids*) [Pal09].

Digraphs

(2-sorted)

Example 6.1 (Directed Graphs)**.** We begin with the partial Lawvere theory of *directed graphs*, which has a sort O of vertices and a sort A of edges, together with source and target operations:



The associated variety is the category of directed graphs, as model morphisms F must satisfy:

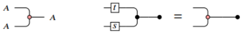
Example 6.2 (Reflexive Graphs)**.** Extending Example 6.1, we ask that each vertex has a self-loop:

then morphisms of models are required to preserve the self-loop, so the associated variety is the category of *reflexive graphs*. Notice that along with Example 6.1, this could also be presented as a (total) 2-sorted Lawvere theory, since all the operations are total.

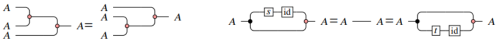
Categories

(2-sorted)

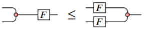
Example 6.3 (Categories). To capture *categories* we extend Example 6.2 with a composition operator, which is defined when the target of the first arrow matches the source of the second:



and equations insisting composition is associative and unital, with identities given by the self-loops:



Model morphisms are precisely functors. It is worth noting that this involves an inequality:

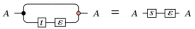


This states that if f and g are composable then so are Ff and Fg, and in particular $F(f \fatsemi g) = Ff \fatsemi Fg$. If this were an equality, it would insist also that if Ff and Fg are composable, then so are f and g, which is not always the case. Of course, the associated variety is the category of small categories.

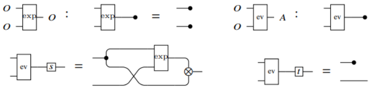
Note the inequality! This accounts for the laxity of natural tns.

Cartesian (closed) categories

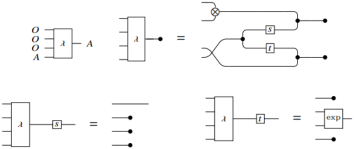
Example 6.8 (Cartesian Categories)**.** To capture *cartesian categories* instead, we can extend Example 6.6 with one equation, ensuring that ε is natural:



Example 6.9 (Cartesian Closed Categories)**.** Finally, to capture *cartesian closed categories* we extend Example 6.8 with an operator $\exp : O \otimes O \to O$, the idea being that $\exp(A, B)$ is the internal hom $[A, B]$, along with an operator $\mathrm{ev} : O \otimes O \to O$ that gives the corresponding evaluation map:

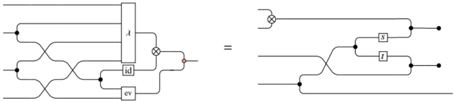


along with an operation λ and equations stating, intuitively, that $\lambda(X, A, B, f)$ is defined precisely in case $f : X \times A \to B$, and yields a map $\lambda(X, A, B, f) : X \to [A, B]$ as in:

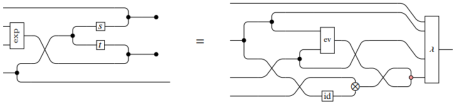


Cartesian (closed) categories

also equations insisting that if $f : X \times A \to B$ then $(\lambda(X, A, B, f) \times 1) \mathbin{;} \mathsf{ev} = f$ holds:



and that if $g : X \to [A, B]$ then $\lambda(X, A, B, (g \times 1) \mathbin{;} \mathsf{ev}) = g$ holds:



Now the associated variety is the category of strict cartesian closed categories and strict cartesian closed functors: these preserve hom-objects and, when $\lambda(X, A, B, f)$ is defined, satisfy $F\lambda(X, A, B, f) = \lambda(FX, FA, FB, Ff)$. This presentation of cartesian closed categories is essentially due to Freyd: a version of it is given immediately after the first appearance of the notion of essentially algebraic theory in [Fre72], albeit somewhat informally, and using very different syntax.

Prospects

Comunque la si giri, (algebraic) theories are monoids

How can we recover a monad-theory correspondence?

As already said, there are problems: one can't expect the category of models of a PLT to be monadic over **Par**.

So?

The more I try, the less I know!

- a formal theory of restriction monads
- cartesian monads on the framed bicategory of polynomial functors
- restriction operads
- ...

Tensor product of theories

Law can be equipped with a canonical symmetric monoidal product operation characterised by the fact that models of $\mathcal{S} \otimes \mathcal{T}$ are

- the $\mathcal{S}$-models in the category of $\mathcal{T}$-models, or equivalently,
- the $\mathcal{T}$-models in the category of $\mathcal{S}$-models:

$$\mathbf{Mod}_{\mathcal{S} \otimes \mathcal{T}} \cong \mathbf{Mod}_{\mathcal{S}}(\mathbf{Mod}_{\mathcal{T}}).$$

Find an analogue of this monoidal structure for partial theories.

Free space for discussion.

Free space for discussion.

Free space for discussion.

Free space for discussion.

Free space for discussion.