The art of $\int$ – notable integrals in Category Theory

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ItaCa Liber I

Plan de l’œuvre

«*I have always disliked analysis*»

P.J. Freyd (Algebraic real analysis)

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Caveat: this wants to be a "light" talk (and partly self-promotion).

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$$\int_C T \xrightarrow{\textit{terminal}} T(X,X) \qquad T(X,X) \xrightarrow[\textit{initial}]{} \int^C T$$

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$$\hom_G(A,B) \cong \lim\left(\hom(A,B) \rightrightarrows \prod_{g \in G} \hom(A,B) \right)$$

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$$\begin{aligned}\int^{(C,D)} T(C,D,C,D) &\cong \int^D \int^C T(C,C,D,D) \\ &\cong \int^C \int^D T(C,C,D,D)\end{aligned}$$

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- They provide analogues for a theory of integrations

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Analysis

Dirac deltas

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Every object of the form $\mathsf{y}C$ is tiny, so it is a functor concentrated on the "point" $C \in \mathcal{C}$. Yoneda lemma says that

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Or dually,

$$\int^C \mathrm{y}C(X) \times FX = \mathrm{y}C \otimes_{\mathcal{C}} F \cong FC$$

(the Dirac δ functor concentrated on $C \in \mathcal{C}$ evaluates functors on points).

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  the exterior derivative.
- Let $C : \mathbf{N}^{\text{op}} \to \text{Mod}(\mathbf{R})$ the functor sending n to (the vector space over) smooth maps $Y \to X$ where Y is closed n-dimensional oriented manifold: $\partial : C_{n+1} \to C_n$ is the geometric boundary.

Stokes' theorem

$$C \otimes \Omega : \mathbf{N}^{\text{op}} \times \mathbf{N} \to \text{Vect}$$

is a functor. There is a map

$$\int : C \otimes \Omega \to \mathbf{R}$$

$$(Y \xrightarrow{\varphi} X, \omega) \mapsto \int_Y \varphi^* \omega$$

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Theorem (Stokes): The square

$$\begin{array}{ccc} C_{n+1} \otimes \Omega_n & \xrightarrow{\partial \otimes 1} & C_n \otimes \Omega_n \\ {\scriptstyle 1 \otimes d}\downarrow & & \downarrow{\scriptstyle \int_n} \\ C_{n+1} \otimes \Omega_{n+1} & \xrightarrow[\int_{n+1}]{} & \mathbf{R} \end{array}$$

is commutative for every $n \in \mathbf{N}$. $\int^n C \otimes \Omega$ is a certain H^0...

Distributions

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A Lawvere distribution is a left adjoint between two toposes;

$$\begin{array}{c}\text{dist. between sheaves on } \mathcal{C}, \mathcal{D} \\ \| \\ \text{profunctors } p : \mathcal{C} \rightsquigarrow \mathcal{D}\end{array}$$

(Dirac distributions over a topos $\mathcal{E}$ are points of that topos (geometric morphisms $p : \mathcal{E} \to$ **Set**); complies with intuition)

Convolutions

Let G be a group; the set of regular functions $f : G \to \mathbf{R}$ has a convolution operation

$$f * g = y \mapsto \int_G fx \cdot g(y - x) d\mu_G$$

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Let $\mathcal{C}$ be a monoidal category; the category $[\mathcal{C}, \textbf{Set}]$ becomes a monoidal category with a convolution operation of presheaves:

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(what if $\mathcal{C}$ is closed? You recover the above formula)

Fourier theory

Fact:

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A profunctor $K : \mathcal{C} \rightsquigarrow \mathcal{D}$ between monoidal categories is a multiplicative kernel if the associated

$$\hat{K} : [\mathcal{C}^{\mathrm{op}}, \mathcal{V}] \leftrightarrows [\mathcal{D}^{\mathrm{op}}, \mathcal{V}]$$

is a strong monoidal adjunction wrt convolution product.

Fourier theory

The K-Fourier transform $f \mapsto \mathfrak{F}_K(f) : \mathcal{D} \to \mathcal{V}$, obtained as the image of $f : \mathcal{C} \to \mathcal{V}$ under the left Kan extension $\mathrm{Lan}_y K : [\mathcal{C}, \mathcal{V}] \to [\mathcal{D}, \mathcal{V}]$.

$$\mathfrak{F}_K(f) : X \mapsto \int^A K(A, X) \otimes fA.$$

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The dual Fourier transform is defined as:

$$\mathfrak{F}^\vee(g) : Y \mapsto \int_A [K(A, X), gA]$$

(prove the relation $\mathfrak{F}^\vee_K(g) \cong \mathfrak{F}_K(g^*)^*$)

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$\mathfrak{F}_\mathrm{y}$ is the identity functor; analogue in analysis, what is the Fourier transform of δ?

Fourier theory

- $\mathfrak{F}_K$ preserves the upper convolution of presheaves f, g, defined as

$$f \mathbin{\overline{*}} g = \int^{AA'} fA \otimes gA' \otimes \mathcal{C}(A \otimes A', -);$$

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dually,

- $\mathfrak{F}_K^\vee$ preserves the lower convolution of presheaves f, g, defined as

$$f \mathbin{\underline{*}} g = \int_{AA'} (fA^* \otimes (gA')^* \otimes \mathcal{C}(A \otimes A', -))^*$$

Fourier theory

Define the pairing $(\mathcal{C},\mathcal{V})\times(\mathcal{C},\mathcal{V})\to\mathcal{V}$ as the twisted form of functor tensor product

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Define the pairing $(\mathcal{C}, \mathcal{V}) \times (\mathcal{C}, \mathcal{V}) \to \mathcal{V}$ as the twisted form of functor tensor product

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If K is a kernel such that $\mathrm{Lan}_y K$ is fully faithful, we have Parseval formula:

$$\langle f, g \rangle \cong \langle \mathfrak{F}_K(f), \mathfrak{F}_K(g) \rangle.$$

Bibliography

- ___________, Coend calculus, London Mathematical Society Lecture Note Series (2020).

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- Yoneda, Nobuo. On Ext and exact sequences. J. Fac. Sci. Univ. Tokyo Sect. I 8.507-576 (1960): 1960.

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- Marta Bunge and Jonathon Funk, Singular coverings of toposes, Lecture Notes in Mathematics vol. 1890 Springer Heidelberg (2006).
- Day, Brian J. Monoidal functor categories and graphic Fourier transforms. Theory and Applications of Categories 25.5 (2011): 118-141.

When you come across a paper with page after page of nothing but enriched categories and coend formulas: