Category.

Able to . unify two common math structures

- monoids
- partially ordered set

A monoid is a set M equipped with

- An element $e \in M$ (in partic M is $\neq \emptyset$)
- A binary operation $\_\cdot\_ : M \times M \longrightarrow M$

$$(m, n) \longmapsto m \cdot n$$

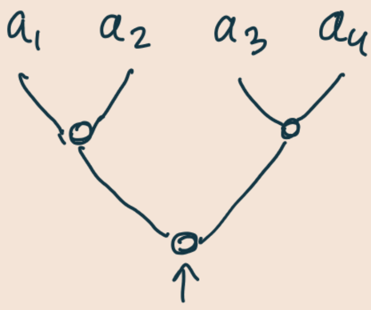
Such that

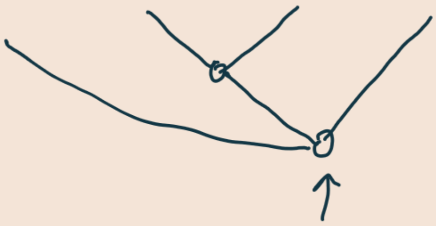
- $\_\cdot\_$ is associative:

$$a, b, c \in M \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(by induction one shows that all bracketings of $a_1 a_2 \_\_ a_n$ are the same element:

$$(a_1 \cdot a_2) \cdot (a_3 \cdot a_4) = a_1 \cdot ((a_2 \cdot a_3) \cdot a_4)$$





- $e \in M$ acts as neutral elem of multiplication

$$m \in M \qquad m \cdot e = m = e \cdot m$$

$(\mathbb{N}, +, 0)$

$(\mathbb{N}, \cdot, 1)$ (group)

$(\mathbb{Z}, +, 0)$, $(\mathbb{Z}, \cdot, 1)$

$\text{List}(\{a, \dots, z\})$ alphabet

$w = (a_1, \dots, a_n)$ $a_i \in$ alphabet

w, u

$\begin{cases} w = azkttb \\ u = rstv \end{cases}$

On the other hand, a partially ordered set ("poset") is a set P over which there is a relation $\leq$ that axiometizes the idea that some elems of P are smaller / bigger than others.

The relation is

1. REFLEXIVE : for every $x \in P$ $\quad x \leq x$
2. TRANSITIVE : for every x, y, z if $x \leq y$ & $y \leq z$ then $x \leq z$
3. ANTISYMMETRIC

(1+2 define what is called a PREORDERED SET)

→ if $x \leq y$ & $y \leq x$ then $x = y$

Over the set {people in this room}

define "$a \leq b$" to mean

"year of birth of a $\leq$ y.o.b. of b"

REFL, TRANSITIVE but not antisymmetric

$(\mathbb{N}, \leq)$ where $\{0 \leq 1 \leq 2 \leq \dots\}$

$(\mathbb{Z}, \leq)$ $\quad \{\dots \leq -2 \leq -1 \leq 0 \leq 1 \leq 2 \leq \dots\}$

$(\mathbb{Q}, \leq)$

$(\mathbb{R}, \leq)$

$\vdots$

all these orders are "total": Given any two elems. x, y either $x \leq y$ or $y \leq x$

An partial order which is not total is :

A any set, $P(A) = \{ U \subseteq A \}$ subsets of A

$P(A)$ is ordered by the inclusion relation

$U \leq V$ in $P(A)$ if $U \subseteq V$ $(\overset{\text{both}}{\subseteq} A)$

Clearly this is a partial order

$(U, V, W \subseteq A)$

- $U \subseteq U$
- $U \subseteq V$ and $V \subseteq W$ then $U \subseteq W$

- $U \subseteq V$ and $V \subseteq U$ $U = V$

Not total

U, V such that $U \not\subseteq V$ and $V \not\subseteq U$

Try if $A = \{ 1, 2, 3, 4 \}$

Different as they may seem, monoids and posets are both instances of categories.

Definition A "category" consist of

Data:

- A collection of objects $\mathcal{C}_0 = \{ X, Y, Z, \dots \}$
- A collection of arrows morphisms $\mathcal{C}_1$

Depicted as " $\underset{\underset{\text{arrow}}{\uparrow}}{f} : \underset{\underset{\text{domain of } f}{\uparrow}}{X} \longrightarrow \underset{\uparrow \text{codomain of } f}{Y}$ "

- To every object $X \in \mathcal{C}_0$ is associated an arrow called the identity of X $1_X : X \longrightarrow X$ $(id_X \dots)$

– Every pair of arrows $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$
(every pair of arrows f, g such that
codomain of f = domain of g
(target) (source)
can be "composed" into an arrow $X \longrightarrow Z$

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "g\circ f"'] & & Z
\end{tikzcd}$$

$$x \longmapsto f(x) \longmapsto g(f(x))$$
$$(g\circ f)(x)$$

All this data is such that:

– Composition is associative
Every time it is possible $\underline{h\circ(g\circ f)} = \underline{(h\circ g)}\circ f$

$$\begin{tikzcd}
& Y \arrow[rr, "h\circ g"] \arrow[dr, "g"] & & W \\
X \arrow[ur, "f"] \arrow[rr, "g\circ f"'] \arrow[urrr, shift left] \arrow[urrr, shift right] & & Z \arrow[ur, "h"'] &
\end{tikzcd}$$

– $$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, equal, "1_X"] \arrow[rr, "f = f\circ 1_X"'] & & Y
\end{tikzcd}$$
$f\circ 1_X = f$ and
$$\begin{tikzcd}
& Y \arrow[dr, equal] & \\
X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$
$1_Y\circ f = f$

for every $f : X \longrightarrow Y$

We will only be interested in "locally small" categories:
for every two objects $X, Y \in \mathbb{C}_0$
$$\mathbb{C}(X,Y) = \{ f : \underset{\text{dom}}{X} \longrightarrow \underset{\text{codomain}}{Y} \}$$ is a set.

Clearly this def is engineered to capture

Set = sets as objects = Set_0
functions as arrows = Set_1
id funct as 1_X
function composition as the comp operation

as an example.
the category of sets

Tweak the definition to talk about partial functions

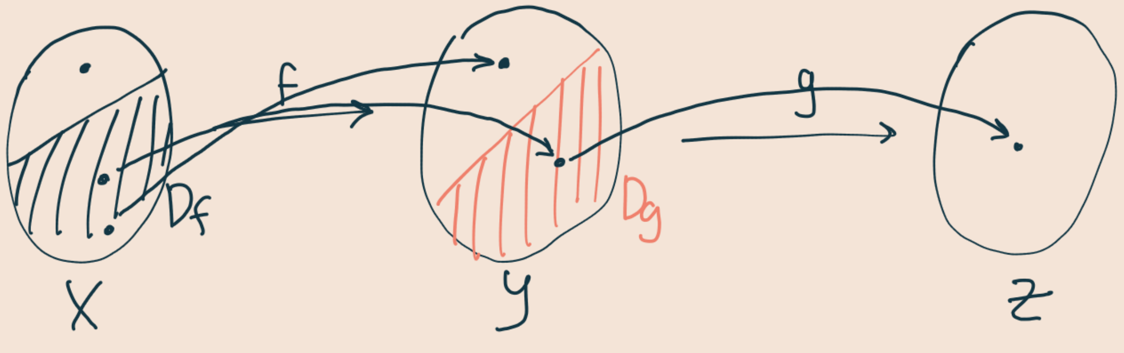
pSet (Par) is the category having

- sets as objects X, Y, Z ....
- partial functions as arrows

partial function := $(D_f \subseteq X$, function $f : D_f \to Y)$
$f : X \dashrightarrow Y$

$\left(\mathrm{head} : \mathrm{List}\, \mathbb{N} \longrightarrow \mathbb{N} \quad D_{\mathrm{head}} = \mathrm{List}\, \mathbb{N} \setminus \{[\,]\} \right)$ empty list

functions are precisely the partial functions where $D_f = X$



if $x \in D_f$ is in D_f, and $f(x) \in D_g$ ☑ compose the function
if $x \in D_f$ but $f(x)$ not in D_g then not defined
if $x \notin D_f$ then not defined.

Identity partial function is the totally defined function $x \longmapsto x$.

So defined, Par is a category.

All algebraic structures (e.g. · vector spaces in linear algebra

- sets functions
- sets partial functions
- graphs in discrete mathem-
- Monoids
- topological spaces, continuous functions
- probability spaces, measurable functions

constitute examples of categories as "universes". Because they are "big".

<u>Otoh</u> a category can also be (very) small.

- $\emptyset$ empty category
- $\{\ast\}$ has a single object $\ast$
  has a single arrow $1_\ast : \ast \longrightarrow \ast$
  composition is $1_\ast \circ 1_\ast = 1_\ast$
- $$\begin{tikzcd} \{0 \arrow[loop above, "1_0"] \arrow[r, "u"'] & 1\} \arrow[loop above, "1_1"] \end{tikzcd}$$
  two objects
  three arrows (2 identities + $u : 0 \rightarrow 1$)
  composition is also boring
  $u \circ 1_0 = u = 1_1 \circ u$
- $$\begin{tikzcd} \{0 \arrow[loop above, "1_0"] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & 1\} \arrow[loop above, "1_1"] \end{tikzcd}$$
  $u \neq v$
  two objects
  4 arrows (2 non-identity)
  u, v are not composable