Universal property of products in a cat.

Let $\mathbb{C}$ be a category. A "product" of two objects $A, B \in \mathbb{C}$ is an object of $\mathbb{C}$, call it P equipped with two arrows

$$A \xleftarrow{\pi_A} P \xrightarrow{\pi_B} B$$

with the property that for all other $Z \in \mathbb{C}_0$ equipped with arrows

$$A \xleftarrow{\alpha} Z \xrightarrow{\beta} B$$

there exists a unique $Z \xrightarrow{\langle \alpha, \beta \rangle} P$ such that

$$\begin{cases} \pi_A \circ \langle \alpha, \beta \rangle = \alpha \\ \pi_B \circ \langle \alpha, \beta \rangle = \beta \end{cases}$$

$\rightsquigarrow$ Any two products of A, B are isomorphic

If $\begin{tikzcd}[row sep=small] & A \\ P \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$ is a product & $\begin{tikzcd}[row sep=small] & A \\ Q \arrow[ur] \arrow[dr] & \\ & B \end{tikzcd}$ is a product

use the universal property of P against Q
use the universal property of Q against P.
use that id_P, id_Q are the only maps making

$$\begin{tikzcd} & A & \\ P \arrow[ur] \arrow[rr] \arrow[dr] & & P \arrow[ul] \arrow[dl] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} & A & \\ Q \arrow[ur] \arrow[rr] \arrow[dr] & & Q \arrow[ul] \arrow[dl] \\ & B & \end{tikzcd}$$

commutative. Denote <u>THE</u> product of A, B as $A \times B$.

Define a category $\prod(A,B)$ having

Define a category

objects: $A \xleftarrow{\alpha} Z \xrightarrow{\beta} B$

arrows:
$$\begin{tikzcd} & A & \\ Y \arrow[ur, "\alpha"] \arrow[rr, "\varphi"] \arrow[dr, "\beta"'] & & Z \arrow[ul, "\alpha'"'] \arrow[dl, "\beta'"] \\ & B & \end{tikzcd}$$

Then $A \times B$ is the terminal object of $\Pi(A,B)$. □

Alternative (and equivalent) definition,

$A \times B$ is such that

$$\mathbb{C}(Z, A\times B) \cong \mathbb{C}(Z, A) \times \mathbb{C}(Z, B)$$

naturally in Z; the only thing to prove is naturality in Z, which amounts to

$$\begin{tikzcd} Z \arrow[d, "h"'] & \mathbb{C}(Z, A\times B) \arrow[r, "\langle \pi_A \circ -, \pi_B \circ - \rangle"] & \mathbb{C}(Z, A) \times \mathbb{C}(Z, B) \\ W & \mathbb{C}(W, A\times B) \arrow[u, "- \circ h"'] \arrow[r, "\langle \pi_A \circ -, \pi_B \circ - \rangle"'] & \mathbb{C}(W, A) \times \mathbb{C}(W, B) \arrow[u, "(- \circ h) \times (- \circ h)"'] \end{tikzcd}$$

$$\begin{tikzcd} u \circ h \arrow[r, mapsto] & \langle \pi_A u h, \pi_B u h \rangle \\ u \arrow[u, mapsto] \arrow[r, mapsto] & \langle \pi_A u, \pi_B u \rangle \arrow[u, mapsto] \end{tikzcd}$$

□

□ Examples of products in various cats.

In the category of • sets

• groups

- vector spaces
- partial orders
- graphs

☐ Examples of categories where not all products exist

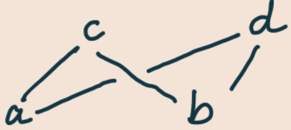
- category of fields (what is the characteristic of $A \times B$?)

- Finite sets and bijections

$$\{1,2\} = A \xleftarrow{\text{bij}} A \times B \xrightarrow{\text{bij}} \{1,2,3\}$$

would imply the presence of a bijection $\{1,2\} \simeq \{1 2 3\}$

- A poset without $\wedge$ for all $x, y \in P$

for example 

: $c \wedge d$ does not exist

$c \wedge d \leq c$ & $\forall z : \begin{matrix} z \leq c \\ z \leq d \end{matrix}$ $z \leq c \wedge d$
$c \wedge d \leq d$

set of lower bounds of c and $d \equiv \{a, b\}$: doesn't have a greatest element

- A monoid M regarded as a category $\{ \bullet \circlearrowright m \mid m \in M \}$

The product of $\bullet$ and $\bullet$ can only be $\bullet$

$$\bullet \xleftarrow{p} \bullet \xrightarrow{q} \bullet$$

There must exist $p, q \in M$ such that for all $x, y \in M$

$\exists! \langle xy \rangle \in M$

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, "x"'] \arrow[r, "y"] \arrow[d, dashed, "! \langle x,y \rangle"] & \bullet \\
& \bullet \arrow[ul, "p"] \arrow[ur, "q"'] &
\end{tikzcd}$$

$p \circ \langle x y \rangle = x$

$p \cdot \langle xy \rangle = x$

$q \cdot \langle xy \rangle = y$ $\qquad\qquad x, y$

e.g. in $(\mathbb{N}, \cdot)$ this means that for every pair of natural num there exist p, q (always the same) and a unique n_{xy} such that $p \cdot n_{xy} = x$ & $q n_{xy} = y$

Now if $x = y$, $p\, n_{xx} = x = q\, n_{xx} \to p = q$

So $\forall\, xy$ $\quad p \cdot n_{xy} = x$

$\qquad\qquad p \cdot n_{xy} = y$ $\quad ϟ$

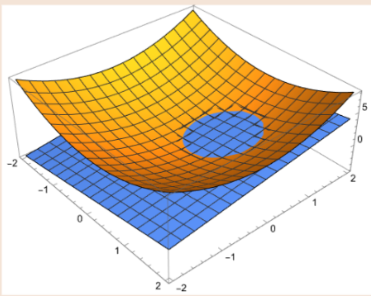
☐ Other shapes of limits (equalizers & pull backs)

Notion of equalizer of two functions

$$E \longrightarrow X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

$E = \{ x \in X \mid f(x) = g(x) \}$ is the subset of "solution to an equation, $f = g$ in this case.

Take $X = \mathbb{R}^2$, $Y = \mathbb{R}$

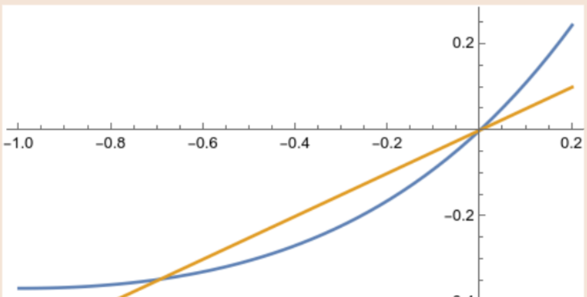


$$f(x,y) = x^2 + y^2$$
$$\overset{?}{=}$$
$$g(x,y) = x + y + 2$$

Take $X = \mathbb{R} = Y$ $\qquad f : \mathbb{R} \longrightarrow \mathbb{R}$

$\qquad\qquad\qquad\qquad g : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = x e^x$$
$$\overset{?}{=}$$
$$g(x) = x/2$$



$f(x) = e^x$

$g(x) = 0$ $\quad = ?$

E is the "equalizer" of f,g; it's the largest subset of X where the proportion ("predicate") «f = g» is true.

This means it satisfies the following univ. prop

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
& A \arrow[u, "q"] \arrow[ul, dashed, "\exists!\, \bar{q}"] &
\end{tikzcd}$$

Coming Soon:

- ☐ Coproducts, coequalizers, pushouts
- ☐ The general notion of limit of a diagram
- ☐ The general notion of colimit of a diagram