• Graphs as functors

Define category $\{1 \rightrightarrows \emptyset\}$

two objects $\emptyset, 1$

2 identity arrows

2 non identity arrows, parallel

A functor $G : \{1 \rightrightarrows \emptyset\} \longrightarrow \mathbf{Set}$ consists of

- Two sets $G1$, $G\emptyset$
- Two functions $G1 \underset{t}{\overset{s}{\rightrightarrows}} G\emptyset$

Interpret this as

$G1$ = set of edges

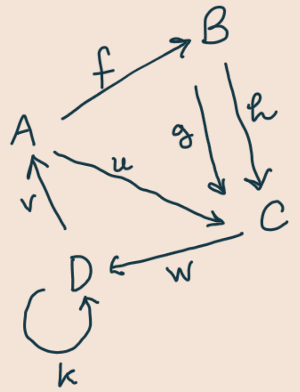
$G\emptyset$ = set of vertices

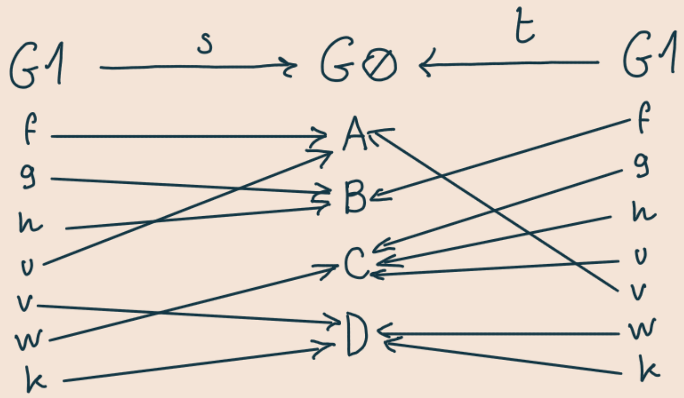
$\Rightarrow$ $G$ is a (directed, multi) graph

$s, t$ act as source & target functions

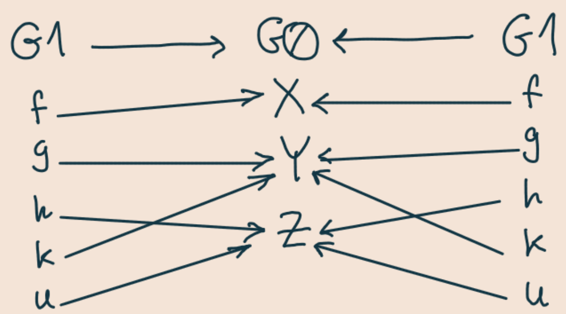
sending an edge to its head / tail

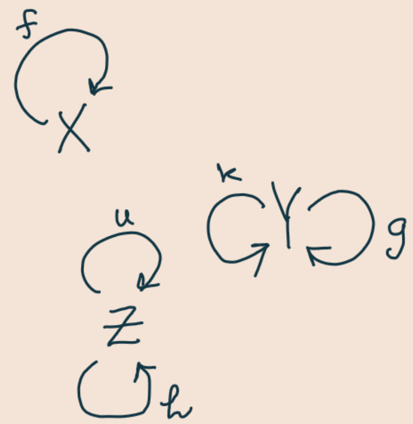
From graph to functor:




From functor to graph

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A graph homomorphism consists of a pair of functions "compatible with source & target" in the sense prescribed by natural transformations

Given graphs $G = (G0, G1, s^G, t^G)$ and $H = (H0, H1, s^H, t^H)$

A graph homomorphism consists of

$f_0 : G0 \longrightarrow H0$

$f_1 : G1 \longrightarrow H1$

take the

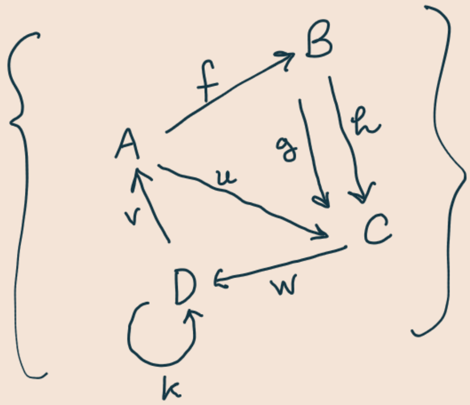
$$\begin{tikzcd} G1 \arrow[r, "f_1"] \arrow[d, "s_G"'] & H1 \arrow[d, "s_H"] \\ G0 \arrow[r, "f_0"'] & H0 \end{tikzcd} \qquad \begin{tikzcd} G1 \arrow[r, "f_1"] \arrow[d, "t_G"'] & H1 \arrow[d, "t_H"] \\ G0 \arrow[r, "f_0"'] & H0 \end{tikzcd}$$

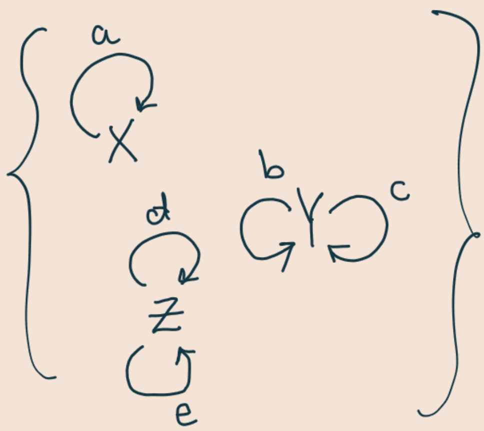
This means that

$$f_1\left(x \xrightarrow{e} y\right) = f_0 x \xrightarrow{f_1 e} f_0 y$$

Is there a graph homomorphism:

G




H

$f_0$

$A \longrightarrow X$

$B \longrightarrow Y$

$C \longrightarrow Z$

$D \longrightarrow Z$

$f_1$

| | |
|---|---|
| f | a |
| g | b |
| h | c |
| k | d |
| u | e |
| v | |
| w | |

? How to solve the issue?

<u>Solution:</u> $f_0$ has to be constant in $X$, $Y$ or $Z$

Say $f_0(\#) = Y$ for all $\# \in \{A, B, C, D\}$ then

$f_1(f) = f_1(g) = f_1(u) = c$;

$f_1$(all the rest) $= b$.

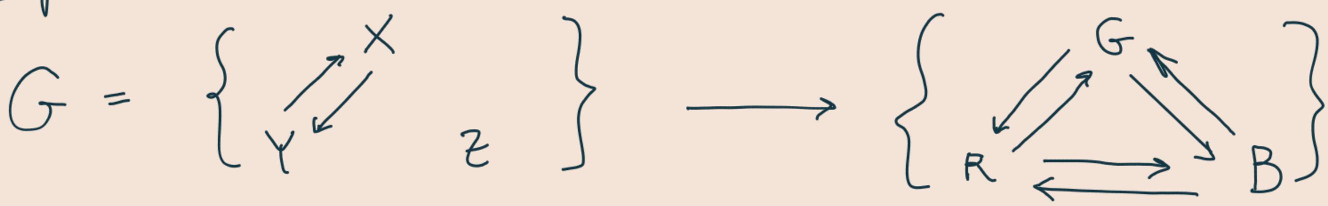
Is a graph homomorphism $f\_G \longrightarrow H$

Fact: a graph homomorphism $G \longrightarrow H$ consists precisely of a natural transformation $G \Rightarrow H$ when $G, H$ are considered functors $\{1 \rightrightarrows \emptyset\} \Rightarrow$ Set.

Naturality is precisely the condition that

$$\begin{tikzcd}
G1 \arrow[r, "f_1"] \arrow[d, shift right, "s^G"'] \arrow[d, shift left, "t^G"] & H1 \arrow[d, shift right, "s^H"'] \arrow[d, shift left, "t^H"] \\
G\emptyset \arrow[r, "f_0"'] & H\emptyset
\end{tikzcd}$$

(Guided) exercise. Determine what is an homomorphism of graphs




Answer: it is a <u>coloring</u> of the graph, i.e. a labeling of the set $\{X, Y, Z\}$ of vertices of a color $\begin{pmatrix} R \text{ red} \\ G \text{ green} \\ B \text{ blue} \end{pmatrix}$ so that when two vertices are connected by an edge, they don't have the same color.

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The topos of graphs is the category of functors $\{1 \rightrightarrows 0) \to$ Set + natural transformations

<u>Exercise</u> A Labeled Transition System is a ~~function~~ set $X$ plus

$X \times \mathbb{N} \times X \xrightarrow{\sigma} \underset{\text{Booleans}}{\{t, f\}}$ ; denote " $x \xrightarrow{n} y$ " if

$\sigma(x, n, y) = \text{true}$.

What is an homomorphism of LTS?

Can one represent a LTS as a graph?

Answer: $(X, \sigma) \rightsquigarrow \begin{pmatrix} X0 = X \\ X1 = \sum h(x,y) \text{ where...} \end{pmatrix}$

## . Reflexive graphs as functors

One can play a similar game with <u>reflexive</u> graphs

Define a category

$$\mathcal{R} := \left\{ \begin{tikzcd} 1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \emptyset \arrow[l, "i" description] \end{tikzcd} \right\} \quad \text{having}$$

- Two objects $\emptyset$ and $1$;
- 2 identity arrows for $\emptyset$ and $1$
- Three non identity arrows $s, t, i$

  subject to the condition $s \circ i = t \circ i = id_{\emptyset}$.

<u>Fact</u>: A functor $\mathcal{R} \longrightarrow Set$ is a special kind of graph $G : \{ 1 \overset{s}{\underset{t}{\rightrightarrows}} \emptyset \}$ + additional condition that

1) Every vertex has an edge $i(v)$ (existence of $i$)
2) the edge $i(v)$ has head $v$, tail $v$.

Every vertex is equipped with a loop $i(v)$

$$\begin{tikzcd} & Y \arrow[loop right, "i_Y"] \\ X \arrow[loop left, "i_X"] \arrow[ur] \arrow[dr] & \\ & Z \arrow[loop right, "i_Z"] \end{tikzcd}$$

A homomorphism of directed graphs is a natural transfe

$$\mathcal{R} \overset{G}{\underset{H}{\rightrightarrows}} Set \quad : \quad (f_0, f_1) : G \Rrightarrow H \quad \text{if} \quad \begin{array}{l} G : \mathcal{R} \Rightarrow Set \\ H : \mathcal{R} \Rightarrow Set \end{array}$$

consists of $f_0 : G\emptyset \to H\emptyset$ compatible with $s, t$ and with $i$

$f_1 : G1 \longrightarrow H1$

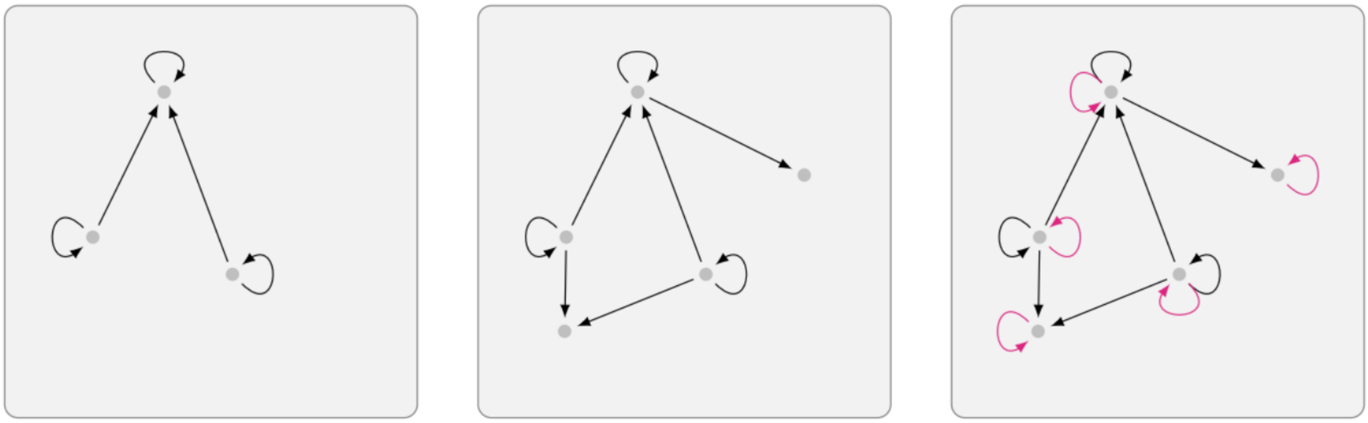
$$\rightsquigarrow \quad f_1\left( \begin{tikzcd} v \arrow[loop left, "i_v"] \end{tikzcd} \right) = \begin{tikzcd} f_0 v \arrow[loop left, "i(f_0 v)"] \end{tikzcd}$$

Every reflexive graph can be regarded as a graph

rGph $\longrightarrow$ Gph is a functor that forgets that a given reflexive graph was reflexive.

Every graph has an associated

- A reflexive envelope where to $\mathcal{G}$ one adds a loop to every vertex (irregardless if there already was one)
- A reflexive core, where one considers the sub-graph over all vertices that have a loop, and defines $i(v)$ accordingly



$\mathcal{G}^r$ $\qquad$ $\mathcal{G}$ $\qquad$ $\tilde{\mathcal{G}}$

core $\subseteq$ graph $\subseteq$ envelope

The topos of reflexive graphs is the category of

$\left\{ \begin{array}{l} \text{functors } \mathcal{R} = \{1 \rightrightarrows 0\} \longrightarrow \mathbf{Set} \ + \\ \text{natural transformations} \end{array} \right.$

. Trees as functors $(\mathbb{N}, \leq)^{op} \longrightarrow Set$

One can represent trees as functors;

Define the category

$$\tau = \{ 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \}$$

poset $\mathbb{N}^{op}$ where $m \longrightarrow n$ if and only if $m \geq n$.

A functor $\tau \longrightarrow Set$ consists of

- A family of sets $X0, X1, X2, \dots X(n)$ for each $n \in \mathbb{N}$
- A family of functions

$$X0 \xleftarrow{f_0} X1 \xleftarrow{f_1} X2 \xleftarrow{f_2} \dots$$

One can represent this as a tree:

$X_0$ = roots

$\uparrow f_0$

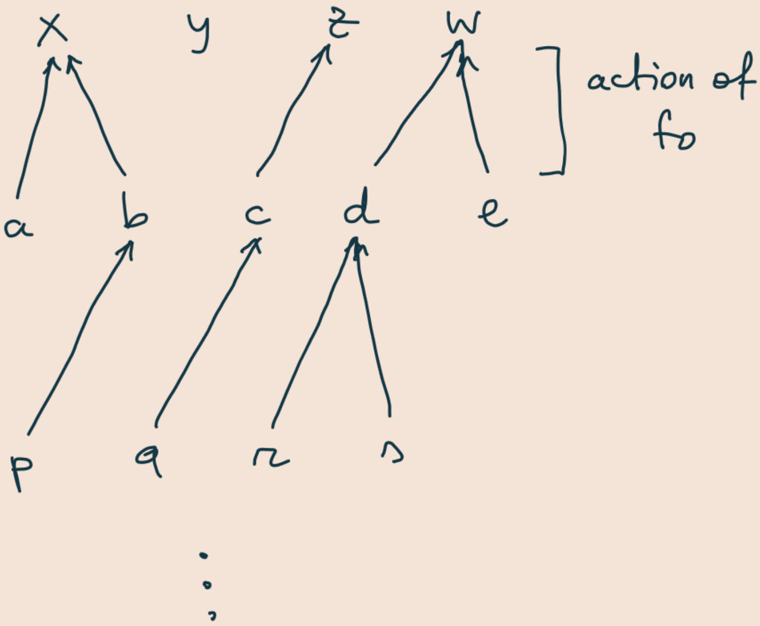
$X_1$ = children @ depth 1

$\uparrow f_1$

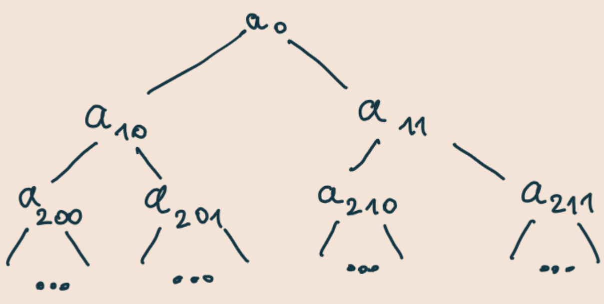
$X_2$ = children @ depth 2

$\uparrow$

$\vdots$




What is the functor associated to the tree




What is the tree associated to the functor

$(n) \longmapsto \{0, 1 \dots n\}$

$\uparrow$ $\qquad \uparrow \sigma : \quad 0 \longrightarrow 0$

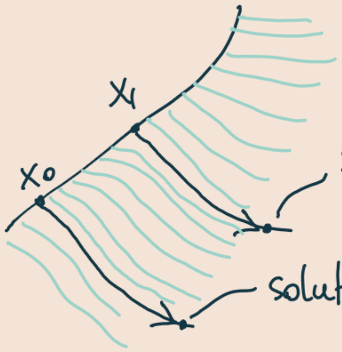
$(n+1) \qquad \{0\ 1 \dots n+1\}^{n+1}$

- Actions as functors $M \longrightarrow \mathrm{Set}$

(e.g. $(\mathbb{R}_{\geq 0}, +)$ : in autonomous ODE, solutions are acted upon by the time translation functions $u(\cdot) \mapsto u(\alpha + \cdot)$

Given an autonomous ODE $\dot{y} = F(y)$ where $F$ doesn't dep on time, $u(t)$ solution $\forall\, \alpha \in \mathbb{R}_{\geq 0}$ $u(\alpha + t)$ solution

Then on the space of solutions acts the monoid $[0, \infty)$




solution to the ODE, initial condition $x_1$

solution to the ODE, initial condition $x_0$

A monoid action consists of

$M \times X \xrightarrow{a} X$ subject to axioms

$a(1, x) = x$

$a(m, a(n, x)) = a(mn, x)$

Fact: A monoid action $a$ is equivalent to a functor $BM \longrightarrow \mathrm{Set}$ where $BM$ is the monoid $M$ regarded as category with a single object $\{\bullet\}$

Lets see that.

On objects $\bullet \longmapsto X$

On arrows $\bullet \circlearrowleft m \longmapsto X \circlearrowleft a(m, -)$

Axioms of functor $\Longleftrightarrow$ axioms of action

Example :

$M = (\mathbb{N}, +, 0)$ $\quad X \circlearrowleft \varphi \circlearrowleft \varphi^2 \circlearrowleft \varphi^3 \cdots$

$M = (\mathbb{N}, \cdot, 1)$ $\quad X \circlearrowleft \varphi_2, \varphi_3, \varphi_7$ $\quad \varphi_n = \varphi_{p_1}^{k_1} \circ \varphi_{p_2}^{k_2} \circ \cdots \circ \varphi_{p_r}^{k_r}$

$M = (\mathrm{List}(A), ++, [\,])$ (for example when $A = \{N, S, W, E\}$)

$M = (\mathbb{Z}/3, +, 0)$ $\quad X \circlearrowleft \varphi$ such that $\varphi^3 = id$
(invertible)

• Relation between reflexive graphs and actions of the "graphic monoid"
(equivalence)

$$\mathcal{R} = \{ 1 \underset{t}{\overset{s}{\rightrightarrows}} \varnothing \} \quad (j : \varnothing \to 1) \qquad s \circ j = t \circ j = id_\varnothing$$

$$\Longrightarrow \quad \begin{array}{l} j \circ s = \sigma \\ j \circ t = \tau \end{array} \text{ are idempotent } \quad \begin{array}{l} \sigma \circ \sigma = \sigma \\ \tau \circ \tau = \tau \end{array}$$

and $\sigma \circ \tau = j \cdot s \cdot j \cdot t = j \cdot t = \tau$

$\tau \circ \sigma = j t j s = j \cdot s = \sigma$ $\qquad$ $\sigma, \tau$ are co-constants

$\{1, \sigma, \tau\}$ with this multiplication law is a monoid called the "graphic monoid" and it is the monoid $M = \mathcal{R}(1,1)$ of endomorphisms of $1$ in the cat $\mathcal{R}$.

<u>Claim</u> the monoid $\mathcal{R}$ acts on every reflexive graph's set of edges (or vertices?)

<u>Claim</u> Given an action $\mathcal{R} \longrightarrow \mathrm{Set}$ one obtains a reflexive graph as follows

$X \underset{\tau}{\overset{\sigma}{\rightrightarrows}} X$ are such that $\qquad \sigma(\sigma x \xrightarrow{x} \tau x) = \sigma x \xrightarrow{\sigma x} \tau x$

$\qquad \tau(\sigma x \xrightarrow{x} \tau x) = \sigma x \xrightarrow{\tau x} \tau x$

Dunque $\begin{array}{l} x, \sigma x \\ x, \tau x \end{array}$ sono edge parallele (?)

invece a me servirebbe $\qquad \begin{array}{l} \sigma\tau = \sigma \\ \tau\sigma = \tau \end{array}$

perché così $\quad \sigma(\sigma x \xrightarrow{x} \tau x) = \sigma x \xrightarrow{\sigma x} \sigma x$

$\qquad \tau(\sigma x \xrightarrow{x} \tau x) = \tau x \xrightarrow{\tau x} \tau x$

$\sigma x \circlearrowright \sigma x$ e $\tau x \circlearrowright \tau x$ ...