

OF LIMS AND SETS

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1. REDUCTION OF LIMITS TO PRODUCTS AND EQUALIZERS

Definition 1.1 (Some terminology). A (small) *diagram* in a category $\mathcal{C}$ is a functor

$$\mathcal{J} \longrightarrow \mathcal{C} \quad (1.1)$$

whose domain is a small category. A *cone* for a diagram D consists of a pair (X, c) where X is an object of $\mathcal{C}$ called *tip* of the cone, and c is a natural transformation $\Delta X \Rightarrow D$ from the constant functor at X ; so c consists of a family of arrows in $\mathcal{C}$,

$$c_j : X \longrightarrow D_j \quad (1.2)$$

the *components* of the cone, such that for every morphism $f : i \rightarrow j$ in $\mathcal{J}$ the triangle

$$\begin{array}{ccc} & X & \\ c_i \swarrow & & \searrow c_j \\ Di & \xrightarrow{Df} & Dj \end{array} \quad (1.3)$$

is commutative. The category of cones for D has

- objects the cones (X, c) for D ;
- arrows $h : (X, c) \rightarrow (Y, c')$ the arrows $h : X \rightarrow Y$ in $\mathcal{C}$, between the tips of the cones such that for every $j \in \mathcal{J}$ the triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ c_j \searrow & & \swarrow c'_j \\ & Dj & \end{array} \quad (1.4)$$

The limit $(\lim D, p_j)$ of a diagram D consists of a terminal object in its category of cones. More than often one calls ‘limit of D ’ the tip of the terminal cone, leaving the maps of the cone implicit *this is almost always harmless but slightly incorrect: the limits is composed of **both** parts*.

If a diagram D has a limit $(\lim D, p_j)$ we say that $\mathcal{C}$ *admits* the limit of D ; if for a fixed $\mathcal{J}$, every $D : \mathcal{J} \rightarrow \mathcal{C}$ has a limit, we say that $\mathcal{C}$ has limits of shape $\mathcal{J}$ or that it has $\mathcal{J}$ -limits; if for every element $\mathcal{J}$ of a subclass $\Phi \subseteq \mathbf{Cat}$ of categories, $\mathcal{C}$ has limits of shape $\mathcal{J}$, we say that $\mathcal{C}$ has limits of shape Φ or that it has Φ -limits. If $\mathcal{C}$ has $\mathbf{Cat}$ -limits, we say that $\mathcal{C}$ is (small-)complete.

Definition 1.2. In particular, a category $\mathcal{C}$ has all products if it has all limits of shape $S^\delta \rightarrow \mathcal{C}$ when S^δ is the discrete category over a set S , and $\mathcal{C}$ has equalizers if it has limits over $\mathcal{J} = \{0 \rightrightarrows 1\}$: this is because

- a diagram of shape S^δ specifies precisely a family of objects X_s in $\mathcal{C}$, one for every $s \in S$; being the terminal cone in this case means that there exists a family of arrows $p_s : \prod_s X_s \rightarrow X_s$ indexed by S with the property that
- a diagram of shape $\{0 \rightrightarrows 1\}$ specifies precisely a pair of morphisms $f, g : D_0 \rightarrow D_1$ in $\mathcal{C}$; being the terminal cone in this case means that there exists.

Theorem 1.3. The category **Set** of sets and functions has all products and all equalizers.

Proof.

- the product of a family of sets $\{X_s \mid s \in S\}$ is the usual Cartesian product $\prod_{s \in S} X_s$, constructed as the set of functions $S \rightarrow \bigcup_s X_s$ with the property that $f(s) \in X_s$. This allows to represent the elements of the set $\prod_{s \in S} X_s$ as S -indexed sequences $(x_s \mid s \in S, x_s \in X_s)$. Evidently, $\prod_{s \in S} X_s$ is equipped with projection maps $p_s : \prod_{t \in S} X_t \rightarrow X_s$ for every $s \in S$, picking the s th element of the S -sequence $(x_s \mid s \in S)$.

The universal property of the product $\prod_{s \in S} X_s$ is spelled as follows:

For every set Z and family of functions $z_s : Z \rightarrow X_s$, there exists a unique $\bar{z} : Z \rightarrow \prod_{s \in S} X_s$ such that $p_s \circ \bar{z} = z_s$.

Define $\bar{z}$ to be the function sending $\zeta \in Z$ to the S -sequence $(z_s \zeta \mid s \in S)$. Clearly this is the only possible definition so that

$$\begin{array}{ccc} & \prod_{s \in S} X_s & \\ \nearrow \bar{z} & \downarrow p_s & \\ Z & \xrightarrow{z_s} & X_s \end{array} \quad (1.5)$$

is a commutative triangle for every $s \in S$.

- the equalizer of a pair of maps $f, g : X \rightarrow Y$ consists of the subset $E = \{x \in X \mid fx = gx\} \subseteq X$; it realizes the universal property

For every $u : Z \rightarrow X$ such that $f \circ u = g \circ u$, there exists a unique $\bar{u} : Z \rightarrow E$ such that u equals the composition $Z \rightarrow E \hookrightarrow X$.

Since E is just a subset of X , the universal property of $\text{eq}(f, g)$ can be rephrased as follows: every $u : Z \rightarrow X$ such that $f(u(z)) = g(u(z))$ for every $z \in Z$ takes values in the subset E , defined above. This is evident, as much as it is evident that E is chosen precisely in order to satisfy this property. $\square$

Lemma 1.4.

If $E \xrightarrow{e} A$ is an equalizer of $A \xrightleftharpoons[f]{g} B$, then the following are equivalent:

- (1) $f = g$,
- (2) e is an epimorphism,
- (3) e is an isomorphism,
- (4) id_A is an equalizer of f and g .

$\square$

Theorem 1.5. The category **Set** of sets and functions has all limits.

Proof. Let $\mathcal{D}$ be a small category and $F : \mathcal{D} \rightarrow \mathbf{Set}$ a functor. For every arrow f in $\mathcal{D}$, we denote $s(f)$ the source and $t(f)$ the target of f ; so, if $f : D \rightarrow D'$, $s(f) = D$, $t(f) = D'$.

We prove that the limit $\lim F$ of F is precisely the equalizer of the pair of maps

$$\prod_{D \in \mathcal{D}_0} FD \xrightleftharpoons[\beta^F]{\alpha^F} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.6)$$

where

- $\alpha^F((x_D \mid D \in \mathcal{D})) = (x_{t(f)} \mid f \in \mathcal{D}_1)$;
- $\beta^F((x_D \mid D \in \mathcal{D})) = (Ff(x_{s(f)}) \mid f \in \mathcal{D}_1)$;

This means two things: if $\lim F$ exists, then it must be the equalizer of that pair; otoh, if that pair (α, β) has an equalizer, then such is the limit of F .

We have to prove that

- (1) There exists a cone

$$\lim F \xrightarrow{\bar{p}} \prod_{D \in \mathcal{D}_0} FD \xrightleftharpoons[\beta]{\alpha} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.7)$$

where e equalizes the pair α, β ;

- (2) such cone is terminal; this will mean two things:
 - the universal property of $\lim F$ entails the universal property of $\text{eq}(\alpha, \beta)$;
 - the universal property of $\text{eq}(\alpha, \beta)$ entails the universal property of $\lim F$.

Thus, there is a unique isomorphism $\text{eq}(\alpha, \beta) \cong \lim F$.

Proving 1. is easy; if $(\lim F, p_D)$ exists, all projections $p_D : \lim F \rightarrow FD$ assemble into a unique map $\bar{p} : \lim F \rightarrow \prod_D FD$ (this is the universal property of $\prod_D FD$). Note in passing that by the lemma above if $\mathcal{D}$ is a discrete category, $\prod_{f \in \mathcal{D}_1} F(t(f)) = \prod_{D \in \mathcal{D}_0} FD$, α, β are invertible and thus $\lim F \cong \prod_{D \in \mathcal{D}_0} FD$, as it should be.

Now, $\bar{p} : \lim F \rightarrow \prod_D FD$ equalizes (α, β) , because the components p_D form a cone: the triangle of sets and functions

$$\begin{array}{ccc} & & FD \\ & \nearrow p_D & \downarrow Ff \\ \lim F & & \\ & \searrow p_{D'} & \\ & & FD' \end{array} \quad (1.8)$$

is commutative, whence the fact that for all $\hat{x} \in \lim F$ and $f : D \rightarrow D'$ in $\mathcal{D}_1$ one has

$$\beta^F(\bar{p}(\hat{x})) = (Ff(p_{s(f)}(\hat{x})) \mid f \in \mathcal{D}_1) = (p_{t(f)}(\hat{x}) \mid f \in \mathcal{D}_1) = \alpha^F(\bar{p}(\hat{x})). \quad (1.9)$$

A similar argument for a general cone $(z : Z \rightarrow FD \mid D \in \mathcal{D}_0)$ proves that this is a cone for F if and only if it equalizes (α, β) ; thus, a cone for F must be a terminal cone wrt the property of equalizing (α, β) ; whence the conclusion. $\square$

More generally, for a category $\mathcal{C}$ to have all limits it is necessary and sufficient that it has all (small) products and equalizers.

Theorem 1.6. The following conditions are equivalent:

- $\mathcal{C}$ has all small products and all equalizers;
- $\mathcal{C}$ is small-complete.

Proof. One reasons in a similar fashion, besides not being allowed to use elements (because in $\mathcal{C}$ objects have no ‘points’!). One direction is evidently true: having all limits implies, a fortiori, having some.

For the converse implication, given a diagram $F : \mathcal{D} \rightarrow \mathcal{C}$ with $\mathcal{D}$ small, one can still build the equalizer of the pair of maps

$$\prod_{D \in \mathcal{D}_0} FD \xrightleftharpoons[\beta^F]{\alpha^F} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.10)$$

where α^F and β^F are induced using the universal property of products; let $u : X \rightarrow Y$ be an arrow in $\mathcal{D}$, and consider the diagrams

$$\begin{array}{ccc} \prod_{D \in \mathcal{D}_0} FD & \xrightarrow{\beta^F} & \prod_{f \in \mathcal{D}_1} F(t(f)) \\ \text{proj}_{s(f)} \downarrow & & \downarrow \text{proj}_u \\ F(s(u)) & \xrightarrow{Fu} & F(t(f)) \end{array} \quad \begin{array}{ccc} \prod_{D \in \mathcal{D}_0} FD & \xrightarrow{\alpha^F} & \prod_{f \in \mathcal{D}_1} F(t(f)) \\ & \searrow \text{proj}_{t(u)} & \downarrow \text{proj}_u \\ & & F(t(u)) \end{array} \quad (1.11)$$

Now consider the equalizer

$$Eq(\alpha^F, \beta^F) \longrightarrow \prod_{D \in \mathcal{D}_0} FD \xrightleftharpoons[\beta^F]{\alpha^F} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.12)$$

and prove that the property of being a limit for F is satisfied by the cone

$$Eq(\alpha^F, \beta^F) \longrightarrow \prod_{D \in \mathcal{D}_0} FD \xrightarrow{\text{proj}_X} FX \quad (1.13)$$

□

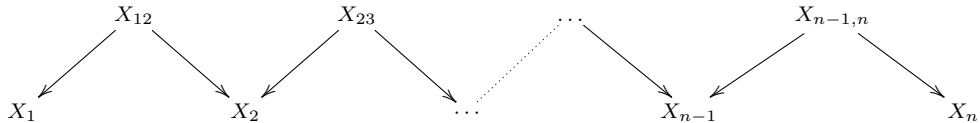
The following question appears in the exercises:

Exercise 1.7. The *triqualizer* $\text{triq}(f, g, h)$ of functions $f, g, h : X \rightarrow Y$ is defined as the limit of the diagram

$$\mathcal{J} = \left\{ 0 \xrightleftharpoons[h]{g} 1 \xrightarrow{f} 1 \right\} \rightarrow \text{Set}. \quad (1.14)$$

- Spell out the universal property of $\text{triq}(f, g, h)$;
- by virtue of Theorem 1.6 above, $\text{triq}(f, g, h)$ must be expressible as an equalizer of two maps between products. How?

Exercise 1.8. An *Urizen compass* G_n is a limit of a diagram of the following form:



write the universal property for G_3 and G_4 ; express G_3 and G_4 as equalizers of maps between products.

2. FUNCTORIALITY, INTERCHANGE OF LIMITS

Let's work in the category of sets.

2.1. Functoriality of limits.

Theorem 2.1 (The 'lim' functor). A natural transformation $\alpha : D \Rightarrow D'$ in the category of functors $D : \mathcal{J} \rightarrow \mathbf{Set}$, induces a morphism $\lim \alpha : \lim D \rightarrow \lim D'$ between the limits. Moreover, $\lim$ preserves identities and compositions, hence it is a functor

$$\lim : [\mathcal{J}, \mathbf{Set}] \longrightarrow \mathbf{Set} \quad (2.1)$$

The natural transformation α has components $Dj \rightarrow D'j$, and the composition $\lim D \xrightarrow{p_j} Dj \xrightarrow{\alpha_j} D'j$ is easily seen to be a cone for D' . Then, there is a unique $\bar{\alpha} : \lim D \rightarrow \lim D'$ such that each diagram

$$\begin{array}{ccc} \lim D & \xrightarrow{\bar{\alpha}} & \lim D' \\ p_j \downarrow & & \downarrow p'_j \\ Dj & \xrightarrow{\alpha_j} & D'j \end{array} \quad (2.2)$$

commutes. Uniqueness proves that $\overline{\beta \circ \alpha} = \bar{\beta} \circ \bar{\alpha}$ and $\overline{\text{id}_D} = \text{id}_{\lim D}$.

Definition 2.2. Let $D : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram; let $F : \mathcal{C} \rightarrow \mathcal{X}$ be a functor; let $\lim D$ denote the limit of D and $\lim(FD)$ the limit of the composite functor $F \circ D : \mathcal{J} \rightarrow \mathcal{X}$; then, the functoriality of F defines a cone

$$\gamma_{F,j} : F(\lim D) \longrightarrow FDj \quad (2.3)$$

whence a unique morphism, in $\mathcal{X}$, $\gamma_F : F(\lim D) \rightarrow \lim(FD)$. We say that F *preserves* $\lim D$, or that it *commutes* with $\lim D$ if γ_F is an isomorphism in $\mathcal{X}$. If this property holds for all diagrams, we say that F preserves all limits, or that F is 'continuous'.

Evidently, F preserves all limits if and only if it preserves all products and all equalizers; it's easy to show that F preserves products and equalizers directly from the definition.

Proposition 2.3. Limits in the category $[\mathcal{C}, \mathbf{Set}]$ are computed objectwise in $\mathbf{Set}$: this means that given a diagram

$$D : \mathcal{J} \longrightarrow [\mathcal{C}, \mathbf{Set}] \quad (2.4)$$

its limit $\lim D$ is a functor $\mathcal{C} \rightarrow \mathbf{Set}$ defined as $(\lim_j D_j)(C) = \lim_j (D_j C)$.

2.2. Limits commute with limits. The following argument will work for any category $\mathcal{C}$ be a category admitting products and equalizers, but we will spell it out just for sets; let S be a set; given a S -indexed family of parallel arrows

$$X_s \begin{array}{c} \xrightarrow{f_s} \\ \xrightarrow{g_s} \end{array} Y_s \quad (2.5)$$

one can consider the parallel maps induced between the products using Theorem 2.1,

$$\prod_s X_s \begin{array}{c} \xrightarrow{(\prod_s f_s)} \\ \xrightarrow{(\prod_s g_s)} \end{array} \prod_s Y_s. \quad (2.6)$$

The map $(\prod_s f_s)$ is defined as the unique with the property that for every $s' \in S$, the square

$$\begin{array}{ccc} \prod_s X_s & \xrightarrow{(\prod_s f_s)} & \prod_s Y_s \\ p_{s'}^X \downarrow & & \downarrow p_{s'}^Y \\ X_{s'} & \xrightarrow{f_{s'}} & Y_{s'} \end{array} \quad (2.7)$$

commutes. Similarly, for $(\prod_s g_s)$.

Theorem 2.4. The equalizer of (2.6) above is the product $\prod_s h_s : \prod_s E_s \rightarrow \prod_s X_s$ of the equalizers $(E_s \xrightarrow{h_s} X_s \rightrightarrows Y_s \mid s \in S)$. In other words, the following ‘Fubini’ formula for the exchange of limits holds:

$$\lim_{0 \rightrightarrows 1} (\lim_{S^\delta} D) \cong \lim_{S^\delta} (\lim_{0 \rightrightarrows 1} D). \quad (2.8)$$

A single function $v : Z \rightarrow \prod_s X_s$ corresponds to a family of functions $(v_s : Z \rightarrow X_s \mid s \in S)$, and the fact that $(\prod_s f_s) \circ v = (\prod_s g_s) \circ v$ is exactly equivalent to the fact that putting $v_s = p_s \circ v$

$$\forall s \in S. (f_s \circ v_s = g_s \circ v_s) \quad (2.9)$$

All in all, every part of the following diagram does commute, for every $s \in S$:

$$\begin{array}{ccc} \prod_s E_s & \xrightarrow{p_s^E} & E_s \\ \prod_s h_s \downarrow & & \downarrow h_s \\ \prod_s X_s & \xrightarrow{p_s^X} & X_s \\ (\prod_s f_s) \downarrow \parallel (\prod_s g_s) & & \downarrow \parallel f_s \downarrow g_s \\ \prod_s Y_s & \xrightarrow{p_s^Y} & Y_s \end{array} \quad (2.10)$$

Now, let’s prove that in these notations

$$(\prod_s f_s) \circ v = (\prod_s g_s) \circ v \iff \forall s \in S. (f_s \circ v_s = g_s \circ v_s). \quad (2.11)$$

Indeed,

- if $(\prod_s f_s) \circ v = (\prod_s g_s) \circ v$, then for all $s \in S$

$$p_s^Y \circ (\prod_s f_s) \circ v = p_s^Y \circ (\prod_s g_s) \circ v \quad (2.12)$$

but the LHS of this equation is $f_s \circ p_s^X \circ v$, and RHS is $g_s \circ p_s^X \circ v$.

- Conversely, if $\forall s \in S. (f_s \circ v_s = g_s \circ v_s)$, then one uses uniqueness: for all $s \in S$,

$$p_s^Y \circ (\prod_s f_s) \circ v = f_s \circ p_s^X \circ v = g_s \circ p_s^X \circ v = p_s^Y \circ (\prod_s g_s) \circ v \quad (2.13)$$

but there exists a *unique* $w : Z \rightarrow \prod_s Y_s$ such that for all $s \in S$ one has that $p_s^Y \circ w$ equals the common value $f_s \circ p_s^X \circ v_s = g_s \circ p_s^X \circ v_s$, hence $w = (\prod_s f_s) \circ v = (\prod_s g_s) \circ v$.

Now, given this, each v_s equalizes (f_s, g_s) , and thus factors through the equalizer $Z \xrightarrow{\bar{v}_s} E_s \xrightarrow{h_s} X_s$ so that $v_s = h_s \circ \bar{v}_s$; by the universal property of $\prod_s E_s$, there is a unique map $\bar{v} : Z \rightarrow \prod_s E_s$, which must be so that $(\prod_s h_s) \circ \bar{v} = v$:

$$\begin{array}{ccc}
 \prod E_s & \xrightarrow{\prod_s h_s} & \prod_s X_s \xrightleftharpoons[(\prod_s g_s)]{(\prod_s f_s)} \prod_s Y_s \\
 \nwarrow \exists! \bar{v} & & \uparrow v \\
 & & Z
 \end{array} \tag{2.14}$$

Exercise 2.5. Let $D : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram; show that if $\mathcal{J}$ has an initial object 0, then $\lim D = D0$.