Lecture 1

7.2.

Dfn of cat. finds similarity / abstracts two diff alg. structures.

monoid // order theory

$m(x,y)$ $x \leq y$

alg. top logic $p \Rightarrow q$

Dfn: Monoid is a Set equipped with

- a binary operation $- \cdot - : M \times M \longrightarrow M$

$(a,b) \longmapsto a \cdot b$ or ab.

- a distinguished element, the identity element,

$e \in M$ (or 1_M, 1)

subject to the following axioms

- $\forall x \quad x \cdot e = e \cdot x = x$
- $\forall xyz \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ~ the order in which you multiply is inconsequential.

Examples: $\cdot\ (\mathbb{N}, +, 0)$ is a monoid so are $(\mathbb{Z}, +, 0)$, $(\mathbb{Q}, +, 0)$

but also $(\mathbb{Q}, \cdot, 1)$ ← commutative

$\cdot$ (lists $[a,b,c]$, concatenate, $(\ )$) non comm $ab \neq ba$ as lists.

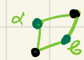
<u>Dfn</u> (Order) $P \in$ Set equipped with a binary relation

subject to the conditions

- $\forall x \in P \quad x \leq x$
- $\forall xyz \in P \quad x \leq y \;\&\; y \leq z \rightarrow x \leq z$

<u>Examples:</u> $(\mathbb{N},\ 0 \leq 1 \leq 2 \cdots)$ total, linear order

A Set, $(\mathcal{P}(A), \subseteq)$ $\quad A \subseteq B \overset{}{\underset{df}{\Leftrightarrow}} x \in A \rightarrow x \in B$.

partial order  $\alpha \nleq \beta \;\&\; \beta \nleq \alpha$

powerset of A. Some disagree that this should be an admissible or elementary operation of Set Theory

LEM + Powerset $\Rightarrow$ Boolean Topos

[nLab] LEM + functions $\rightarrow$ Powerset

(nLab) $\exists P(-) \Leftrightarrow \exists$ functions and set of truth values

They're both antisymmetric though. $x \leq y \;\&\; y \leq x \Rightarrow x = y$

This fails if you order smthing wrt some info while disregarding other info.

eg order ppl wrt birth year.
$\exists$ pair of unequal ppl with same birthyear/day.

Category theory captures the similarity between the two definitions

Collect all finite sets "in a box."

* $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ $\rightarrow$ $A \xrightarrow{g \circ f} B$

* $id_A : A \longrightarrow A$, $a \longmapsto a$ st $f \circ id_A = id_B \circ f = f$

We'd like $((\text{FinSet}, \circ), id_A)$ is a <u>monoid</u>

But here I needed an identity for each object

whereas, in monoids I required a <u>unique</u> identity $e \in M$.

There is a second problem,

Composition of functions is defined only

for consecutive functions

whereas in monoids, one can multiply any pair of elements

(These objections lose its force if we consider $(Aut(A) = \text{functions } A \to A, \circ, id_A)$ which is a monoid)

$$\left[\begin{array}{lll} \text{Monoid} \longrightarrow \text{Category} & & \\ & \xrightarrow[\text{multiple objects}]{\text{add}} & \\ \text{Group} \longrightarrow \text{Groupoid} & & \end{array}\right]$$

*A class is like a very very very big set

<u>Dfn</u> A category $\mathcal{C}$ consists of

- a class* of objects, $A, B, C \cdots \in \mathcal{C}_0$ or $\text{ob}\,\mathcal{C}$
- a class of morphisms, $f, g, h \cdots \in \mathcal{C}_1$ or $\text{mor}\,\mathcal{C}$ (arrows)

Such that:

- Every arrow has unique domain (source), codomain (target): $A \xrightarrow{f} B$, $A = \text{dom}\, f$, $B = \text{cod}(f)$
- Every object has a distinguished identity arrow $\text{id}_X : X \to X$
- Every pair $X \xrightarrow{f} Y \xrightarrow{g} Z \rightsquigarrow X \xrightarrow{g \circ f} Z$

Subject to the following axioms:

i) Identity axiom

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "1_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[dr, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

$$1_Y \circ f = f \qquad f \circ 1_X = f$$

ii) Associativity axiom

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"] \arrow[rr, bend right, "h \circ g"'] & Z \arrow[r, "h"] & W \end{tikzcd}$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Given $X, Y \in \mathcal{C}_0$, we denote with $\mathcal{C}(X,Y)$ the class of arrows "from X to Y"

$$\mathcal{C}(X,Y) = \{ f : X \longrightarrow Y \} = \left\{ f \in \mathcal{C}_1 : \begin{cases} \operatorname{dom} f = X \\ \operatorname{cod} f = Y \end{cases} \right\}$$

Rmks

i) The classes $\mathcal{C}(X,Y)$ for X, Y varying in $\mathcal{C}_0$, are pairwise disjoint.

That's because dom, cod are functions $\mathcal{C}_1 \underset{\text{cod}}{\overset{\text{dom}}{\rightrightarrows}} \mathcal{C}_0$ and hence have uniquely defined outputs.

Therefore, $X \neq Y$ in $\mathcal{C}_0 \Rightarrow 1_X \neq 1_Y$ in $\mathcal{C}_1$

meaning

$$\begin{aligned} \mathbb{1} : \mathcal{C}_0 &\longrightarrow \mathcal{C}_1 \\ X &\longmapsto 1_X \end{aligned}$$

is injective

ii) "Composition of arrows" is a partial function, defined exclusively on "consecutive" arrows

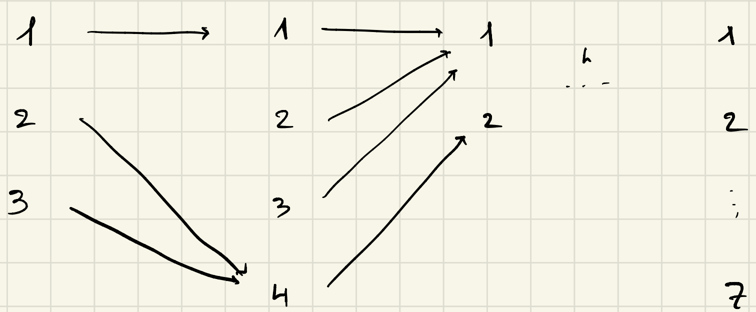
$$\begin{tikzcd} \mathcal{C}_1 \times \mathcal{C}_1 \arrow[rr] & & \mathcal{C}_1 \\ (\mathcal{C}_1 \times \mathcal{C}_1)^{cons} \arrow[u, hook] \arrow[urr] \arrow[r, equal] & \{ (f,g) : \operatorname{cod} f = \operatorname{dom} g \} & \end{tikzcd}$$

Examples:

i) Consider finite sets and functions between them

or better, $\omega = (\{1, \dots, n\})_{n \in \mathbb{N}}$

and functions between them



Every set has an identity function $\forall i \leq n$: $id(i) = i$

Fin has Fin_0 = finite sets of the form

$\{1 \dots n\}$ for $n \in \mathbb{N}$

Fin_1 = functions

Identities as above

Composition of arrows = Comp of funct.

Grp = (Groups , group homs)

Vect = (Vector Spaces , linear functs)

$\vdots$