MONOID ———— M set (nonempty : $1_M \in M$)

$M \times M \longrightarrow M \quad (a,b) \mapsto a \cdot b$

$\begin{cases} a \cdot (b \cdot c) = (a \cdot b) \cdot c & \text{(ASSOC)} \\ 1_M \cdot a = a \cdot 1_M = a & \text{(ID)} \end{cases}$

POSET (partially ordered)

P + binary relation $\leq$ "less or equal"

(REFL) $\forall x \in P \quad x \leq x$

(TRANS) $\forall x, y, z \in P \quad$ if $x \leq y$ & $y \leq z$ then $x \leq z$.

Category simultaneously generalizes monoid and ordered set.

$\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ collections of elements (in general too big to be sets)

$\begin{cases} \mathcal{C}_0 = \text{class of objects} \\ \mathcal{C}_1 = \text{class of arrows / morphisms} \end{cases}$

Each $f \in \mathcal{C}_1$ has a domain, codomain and be drawn as an arrow $\underset{\in \mathcal{C}_0}{\mathrm{dom}(f)} \longrightarrow \underset{\in \mathcal{C}_0}{\mathrm{cod}(f)}$

↳ Allows to represent a cat. as a certain (directed) graph with specified loops, identity arrows

Can compose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$\begin{cases} \mathrm{dom}(g) = \mathrm{cod}(f) \\ \Rightarrow \exists \; g \circ f \text{ composite arrow} \end{cases}$

Operation is not total (def'd only when) but it is still associative & unital

$h \circ (g \circ f) = (h \circ g) \circ f$

$$\begin{tikzcd} A \arrow[d, "1_A"'] & \\ A \arrow[r, "f"] & B \arrow[d, "1_B"] \\ & B \end{tikzcd} \Rightarrow \quad f \circ 1_A = f = 1_B \circ f$$

Remark Given any category $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, d, c, \overset{\circ}{\mathrm{comp}})$

Fix an object $A \in \mathcal{C}_0$.

Then the (class/set) $\mathcal{C}(A,A) = \{A \xrightarrow{f} A\}$ is a monoid

Operation : composition of arrows $(g, f) \mapsto g \circ f$

$g \circ f \quad A \xrightarrow{f} A \xrightarrow{g} A$

Identity : $A \xrightarrow{1_A} A$ id arrow of A

2 axioms

$h \circ (g \circ f) = (h \circ g) \circ f$ because assoc axiom valid in the whole $\mathcal{C}$

$1_A \circ f = f = f \circ 1_A$ again

A category can be thought as a monoid with many objects (each of which determines a monoid according to what we just proved)

A monoid also gives rise to a category, in the following way

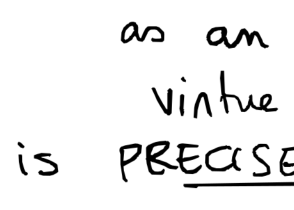
Define a category out of the monoid $(M, \cdot, 1_M)$ $\left(\frac{\text{assoc}}{\text{unital}}\right)$

The class of objects is very small : it is a single element

$\begin{cases} \mathcal{C}_0^M := \{\star\} \\ \mathcal{C}_1^M := \text{the set of elements of } M \end{cases}$

$\mathcal{C}^M$ is a category : composition is the monoid operation which is associative, and has 1_M as an identity PRECISELY by virtue of the monoid axioms

Moral : A monoid is PRECISELY a category of the form



specify a way to compose loops

- associative
- unital 1_M

Now I will show that every $(P, \leq)$ gives rise to a categ.

Define a category $\mathcal{C}^P$ as follows

1) Objects $\mathcal{C}_0^P := P$ elements of P
2) To define the arrows, we stipulate that

$\mathcal{C}^P(x,y)$ = set of arrows with domain x codomain y

contains a single element if $\boxed{x \leq y}$ and it is $\varnothing$ otherwise

(Imagine $\mathcal{C}^P(x,y) = \{\boxed{x \leq y}\}$

There has to be an identity

$\mathcal{C}^P(x,x)$ has to contain an element $\{\boxed{x \leq x}\}$ (comes from the REFL property)

$$\mathcal{C}^P(x,y) \times \mathcal{C}^P(y,z) \longrightarrow \mathcal{C}^P(x,z)$$

$$(\boxed{x \leq y}, \boxed{y \leq z}) \longmapsto \boxed{x \leq z}$$ thx to TRANS property

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In analogy with monoids (= categories w single object but possibly very many arrows)

a ordered set is PRECISELY a category with possibly very many objects, but where every $\mathcal{C}(x,y)$ $(x, y \in \mathcal{C}_0)$ has AT MOST one element

$\mathcal{C}(x,y)$ is either empty or has a single element

$\Rightarrow \mathcal{C}_0$ is an ordered set with respect to the relation "$\leq$" defined as

$$\underbrace{\overset{\text{dom}}{x} \leq \overset{\text{cod}}{y}}_{\text{proposition}} := \underbrace{\mathcal{C}(x,y) \neq \varnothing}_{\text{proposition}}$$

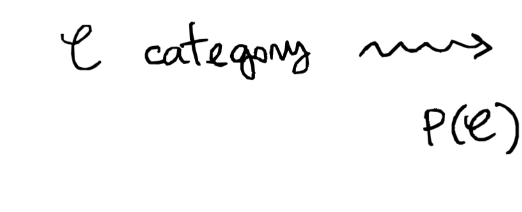
(REFL) $x \leq x$ ☑ $\mathcal{C}(x,x)$ (in the category) $\neq \varnothing$ by the identity axiom

$\mathcal{C}(x,x) = \{\mathrm{id}_x\}$

(TRANS) $x \leq y$ & $y \leq z \rightarrow \underline{x \leq z}$

$\mathcal{C}(x,y) \times \mathcal{C}(y,z)$

$\{\alpha : x \to y\}$ & $\{\beta : y \to z\} \longmapsto \{\beta \circ \alpha : x \to z\}$ ■



CONSTRAINT : $\mathcal{C}(x,y)$ is either $\varnothing$ or has 1 elem

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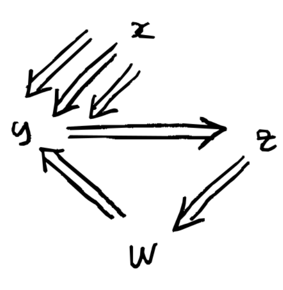
$\mathcal{C}$ category ⇝ posetal reflection of $\mathcal{C}$

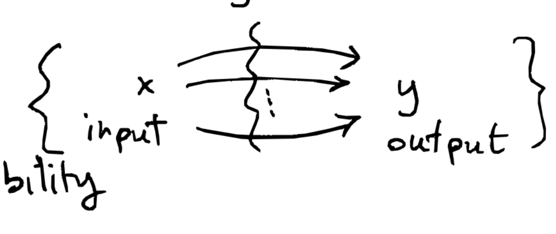
$P(\mathcal{C}) = (\underset{\text{objects}}{\mathcal{C}_0}, \leq)$

↳ $x \leq y \Leftrightarrow \mathcal{C}(x,y) \neq \varnothing$

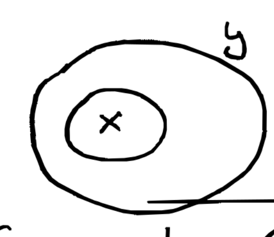
TRANSITION SYSTEM



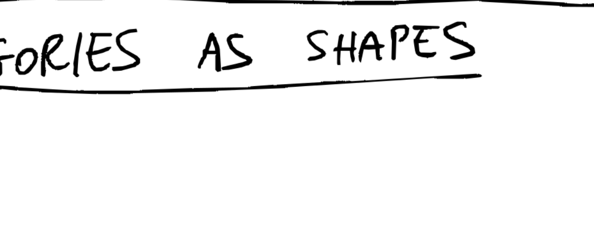
$E = \{\text{states}\}$



"$\Rightarrow$" reachability



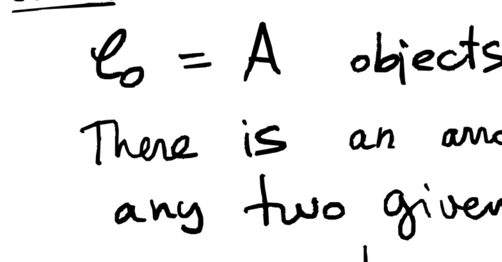
CATEGORIES AS SHAPES	CATEGORIES AS UNIVERSES	CATEGORIES AS STRUCTURES
A category generalizes directed graphs (but also specializes it to having comp, id + axioms...)	A category is a generalized universe to do mathematics (Sets, all functions) is a category (Sh(X))	A category is a simultaneous gen. of - Monoid - Ordered set - ~ The course so far



EXAMPLES of CATEGORIES AS SHAPES

Simplest example

- EMPTY CATEGORY no objects $\mathcal{C}_0 = \varnothing$; no morphisms $\mathcal{C}_1 = \varnothing$
- UNIT CATEGORY has a single object $\{\bullet\} = \mathcal{C}_0$; $\{\bullet \circlearrowleft 1_\bullet\} = \mathcal{C}_1$; $1_\bullet \circ 1_\bullet = 1_\bullet$
- DISCRETE CATEGORY (ON THE SET A)
  - $\mathcal{C}_0 = A$
  - $\mathcal{C}_1 = \{1_a$ identity on the object $a \in A\}$



composition can only happen btwn two identities

This fixes the structure

$1_a \circ 1_b$ iff $a = b$ and must be $1_a = 1_b$

(CODISCRETE)

- CHAOTIC CATEGORY ON A SET A:

$\mathcal{C}_0 = A$ objects

There is an arrow (precisely one) connecting any two given objects :

$\mathcal{C}^A(x,y) = \{u_{xy}\}$ "unique" arrow $x \longrightarrow y$

$$\begin{tikzcd} x \arrow[r, "u_{xy}"] \arrow[rr, bend right, "u_{xz}"'] & y \arrow[r, "u_{yz}"] & z \end{tikzcd}$$

u_{xz} must be the composition $u_{yz} \circ u_{xy}$

with this def

$$u_{wz} \circ (u_{yz} \circ u_{xy}) = (u_{wz} \circ u_{yz}) \circ u_{xy}$$

$\begin{cases} u_{yy} \circ u_{xy} = u_{xy} \\ u_{xy} \circ u_{xx} = u_{xy} \end{cases}$ ~~$u_{xx} \circ u_{yx} = u_{yx}$~~ ~~$u_{yx} \circ u_{yy} = u_{yx}$~~ same on the other side

☑