DEF of CATEGORY

Monoids are categories : exactly a cat with one object
Ordered sets are " : exactly a cat where $x \to y$ at most one

- EMPTY CATEGORY no objects, no morphisms
- TERMINA/UNIT CATEGORY $\bullet \circlearrowleft 1.$
- DISCRETE A, just identites for every $a \in A$ object
- CODISCRETE/CHAOTIC A objects

$$\begin{tikzcd} x \arrow[r, "u_{xy}"] \arrow[dr, "u_{xz}"'] & y \arrow[d, "u_{yz}"] \\ & z \end{tikzcd}$$

y (exactly one)

A^δ

$\underbrace{A^\chi}_{\text{many objects}}$ is equivalent to $\bullet \circlearrowleft 1$

GENERIC ARROW ("WALKING" ARROW)

$^{id}\circlearrowright 0 \xrightarrow{u} 1 \circlearrowleft^{id}$ (cat associated $\{0 \le 1\}$)

More generally the cat associated to the linear order $\{0 \le 1 \le 2 \le \dots \le n\}$ is called generic chain.

$n+1$ objects
unique arrow $i \to j$ anytime $i \le j$

$i \longrightarrow j$ decomposes as
$i \to i+1 \to i+2 \to \cdots \to j$

$\{0 \circlearrowleft\}$ $\quad$ $\{0 \to 1\}$

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[r] & 3 \\ 0 \arrow[ur] \arrow[rr] \arrow[urr] & & 2 \arrow[u] \end{tikzcd}$$

$[0]$ $\quad$ $[1]$ $\quad$ $[2]$ $\quad$ $[3]$

" simplices" of dimension $[0], [1], [2], [3]$

GENERIC DOUBLE ARROW

$\{ \circlearrowright 0 \underset{v}{\overset{u}{\rightrightarrows}} 1 \circlearrowleft \}$ Composition can be defined in only one way

Other examples of cats arising from orders:

GENERIC SQUARE (COMMUTATIVE)

$$\begin{tikzcd} 00 \arrow[r] \arrow[d] \arrow[dr] & 10 \arrow[d] \\ 01 \arrow[r] & 11 \end{tikzcd}$$

Notice that $\square$ arises from the ordered set of pairs $\{(00), (10), (01), (11)\}$ wrt. $(ij) \le (mn)$ if both $i \le m$, $j \le n$

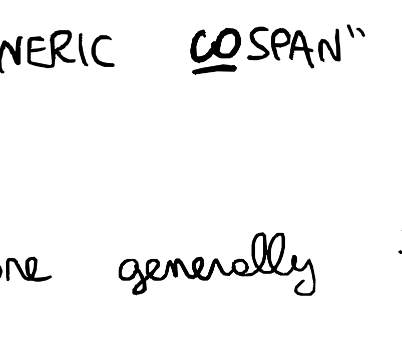
"product order" on $[1] \times [1]$
(tiny example of the product of two categories)

n-DIMENSIONAL CUBE

Note $\square$ is the ordered set of subsets of $\{a, b\}$

$$\begin{array}{ccc} {}_{00}\emptyset & \subseteq & {}_{01}\{b\} \\ \cap\mid & \subseteq & \cap\mid \\ {}_{10}\{a\} & \subseteq & {}_{11}\{a,b\} \end{array}$$

More generally $X = \{x_1 \_\_ x_n\}$ the cube of dimension n is the category assoc. to the powerset of X



dim 4 draw a 4 dim cube.

"GENERIC SPAN"
3 objects

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 1 & & 2 \end{tikzcd}$$

"GENERIC COSPAN"

$$\begin{tikzcd} 1 \arrow[r] & 0 & 2 \arrow[l] \end{tikzcd}$$

More generally the generic S-span is the category obtained as "$S^\triangleleft$"

OBJECTS $\underset{\text{a set}}{S} + \{-\infty\}$

$$\begin{tikzcd} & s \\ -\infty \arrow[ur, "one"] \arrow[r, "one"] \arrow[dr, "one"'] & s' \\ & s'' \end{tikzcd}$$

S

GENERIC ARROW $= \{1\}^\triangleleft$
GENERIC SPAN $= \{1,2\}^\triangleleft$

the generic S-cospan has
OBJECTS $S + \{+\infty\}$

$$\begin{tikzcd} s \arrow[dr, "one"] & \\ s' \arrow[r, "one"] & +\infty \\ s'' \arrow[ur, "one"'] & \end{tikzcd}$$

no NONIDENTITY morphisms $s \leftrightarrows s'$

THE CATEGORY OF "DIGITAL CIRCUITS"

OBJECTS are natural numbers $\{0, 1, 2, \dots\}$
SET of morphisms $n \longrightarrow m$ is the set of functions
$f : \mathbb{B}^n \longrightarrow \mathbb{B}^m$ where $\mathbb{B} = \{t, f\}$ "booleans"

[f is a function accepting n bits of input, giving m bits of outputs

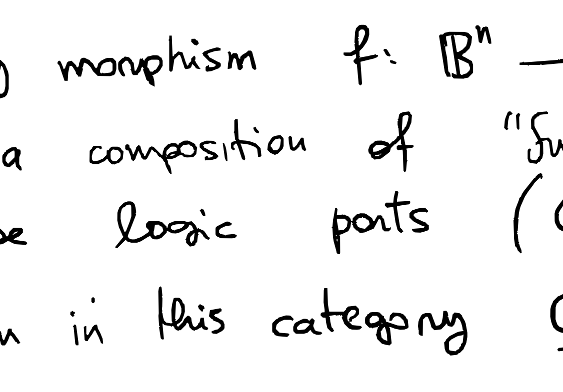
Each such f is uniquely determined by 1[st] component, 2[nd] ", ⋮ m[th] component

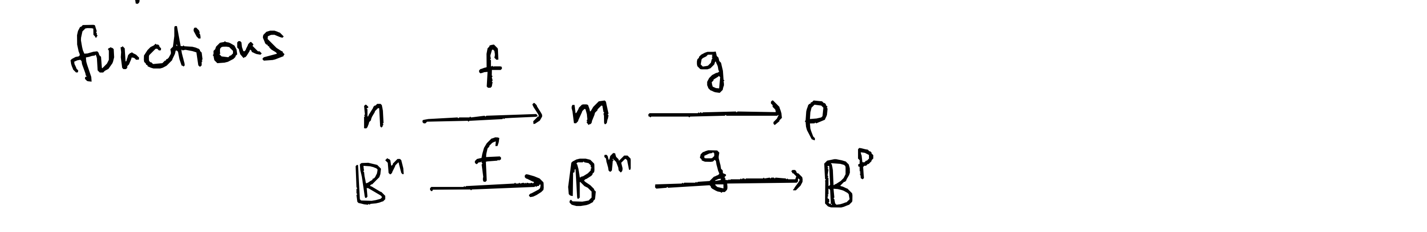
$$f(x_1 \_\_ x_n) = \begin{pmatrix} f_1(x_1 \_\_ x_n) \\ f_2(x_1 \_\_ x_n) \\ \vdots \\ f_m(x_1 \_\_ x_n) \end{pmatrix} \quad x_i \in \mathbb{B}$$

determine f uniquely

So: it's enough to study f's with $\mathbb{B}^1$ "1" as codomain

So: One can represent an $f : n \longrightarrow 1$ as a certain "box" or "gate"



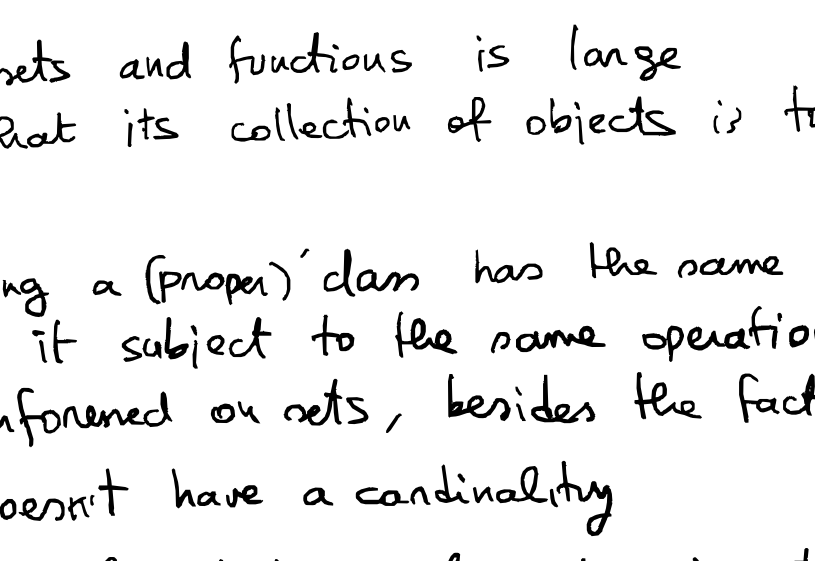


(!) Every morphism $f : \mathbb{B}^n \longrightarrow \mathbb{B}^m$ can be decomposed as a composition of "fundamental" ones, precisely these logic ports (C. Shannon)

Composition in this category Circ is composition of functions

$$\begin{array}{c} n \xrightarrow{f} m \xrightarrow{g} p \\ \mathbb{B}^n \xrightarrow{f} \mathbb{B}^m \xrightarrow{g} \mathbb{B}^p \\ n \longrightarrow p \end{array}$$

$3 \longrightarrow 3$
x, y, z $\quad$ x, y, not z



For a while now the examples of categories will be "large"

- the category of sets and functions is large in the sense that its collection of objects is too large to be a set

Informally speaking a (proper) class has the same properties of a set, and it subject to the same operations that can be performed on sets, besides the fact that

→ 1) A class doesn't have a cardinality
→ 2) the collection of subclasses of a class is not a class

A, B classes $\quad A \times B$ product
A, B classes $\quad A \longrightarrow B$ function btwn classes

↳ TERMINOLOGY

locally small category can have a proper class of obj but every $\mathcal{C}(X, Y) = \{f : X \to Y\}$ is a set

$\mathcal{C}_1$ = class of all arrows decomposes as $\coprod_{X, Y \in \mathcal{C}_0 \times \mathcal{C}_0} \mathcal{C}(X, Y)$

▸ Small if it has a set of objects (thus also a set of morphism)

[Axioms of category do not prevent from costructing a "category" with a set of objects (say, even finite) but where $\mathcal{C}(x, y)$ can be a class. But no one does it!]

The usual understanding is that "category" is short for "locally small" category

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Examples of such locally small categories

▷ Sets and functions ☑

▷ A bigger category in which we think sets are embedded of RELATIONS: (Rel)

objects = all sets
arrows $\mathrm{Rel}(X, Y) = \{R \subseteq X \times Y\}$ subsets of Cart product

Identity of a set X is the relation
$\Delta_X \subseteq X \times X \quad \Delta X = \{(x, x) \mid x \in X\}$ "diagonal" rel'n

A, B, C sets $\quad R \subseteq A \times B \quad S \subseteq B \times C$
$\qquad R : A \longrightarrow B \quad S : B \longrightarrow C$

composition: $\{S \circ R : A \longrightarrow C\} \subseteq A \times C$

$(a, c) \in (S \circ R) \iff \left\{ \exists b \in B \text{ such that } \begin{matrix} (a, b) \in R \\ (b, c) \in S \end{matrix} \right\}$

i) $\Delta \circ R = R$
ii) $R \circ \Delta = R$ } identity L, R
iii) $(R \circ S) \circ T = R \circ (S \circ T)$ } assoc

Show that two sets are equal:
i) $\Delta \circ R \subseteq R$
$R \subseteq \Delta \circ R$

Inside Rel, functions can be characterized as functional relations:

$f \subseteq A \times B$ defines a function $A \longrightarrow B$ if
|| for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$

In such a situation we denote $b = f(a)$ image of a under f.