$\alpha : F \Rightarrow G$ natural

is a <u>natural isomorphism</u> iff all its components are iso (in the codomain of F, G)

$$\begin{tikzcd} \mathcal{C} \arrow[d, bend right, "F"'] \arrow[d, bend left, "G"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ \mathcal{D} \end{tikzcd}$$

$\alpha_c : FC \longrightarrow GC$ are morphisms in $\mathcal{D}$

Observe that $\alpha_c^{-1} : GC \longrightarrow FC$ (inverses of α_c) $\quad \alpha^{-1} : G \Longrightarrow F$ is <u>natural</u>

$$\begin{tikzcd} FC \arrow[r, "Fu"] \arrow[d, "\alpha_c"] \arrow[dr, phantom, "//"] & FC' \arrow[d, "\alpha_{c'}"] \\ GC \arrow[r, "Gu"'] \arrow[u, bend left, "\alpha_c^{-1}"] & GC' \arrow[u, bend left, "\alpha_{c'}^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} c \arrow[d, "\forall u"'] \\ c' \end{tikzcd}$$

$$\alpha_{c'} \circ Fu = Gu \circ \alpha_c$$

$$\underbrace{\alpha_{c'}^{-1} \circ \alpha_{c'}}_{1} \circ Fu \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gu \circ \underbrace{\alpha_c \circ \alpha_c^{-1}}_{1}$$

$$1 \circ Fu \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gu \circ 1$$

$$Fu \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gu$$ is the naturality condition for α^{-1}

Definition $\mathcal{C}^{op} \xrightarrow{F} Set$ is called a PRESHEAF

Given any object of $\mathcal{C}$, $X \in \mathcal{C}_0$

I can consider the functor $hom(-,X)$ def'd as follows

$$hom(-,X)(A) = hom(A,X) \in Set$$
$$= \{A \xrightarrow{f} X\}$$

for $A \in \mathcal{C}_0$

$u: A \longrightarrow B$ $\qquad \underbrace{hom(u,X)}_{-\circ u} : hom(A,X) \longleftarrow hom(B,X)$

$$\left(A \xrightarrow{f\circ u} X\right) \longleftarrow\!\shortmid \left(f : B \longrightarrow X\right)$$

This is a functor "bc. $\mathcal{C}$ is a category":

$hom(id_A, X) = id_{hom(A,X)}$ $\qquad f \longmapsto f\circ id = f$ by category axiom

$hom(v\circ u, X) = hom(u,X)\circ hom(v,X)$

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dr, "v\circ u"'] & B \arrow[d, "v"] \\ & C \end{tikzcd}$$

$hom(v\circ u, X)(f) = f\circ(v\circ u)$

$hom(u,X)\left(hom(v,X)(f)\right)$

$hom(u,X)(f\circ v)$

$(f\circ v)\circ u$

Def a functor $F : \mathcal{C}^{op} \longrightarrow Set$ with the property that there exists an invertible natural transformation

$$\theta : \hom(-, X_F) \Longrightarrow F$$

for some object $X_F \in \mathcal{C}_0$, is called a REPRESENTABLE functor.

(It is a non-obvious fact that when X_F exists, it is unique.) it's a consequence of "Yoneda lemma"

$$\begin{aligned} Cat &\longrightarrow Set \\ \mathcal{C} &\longmapsto \mathcal{C}_0 \\ \mathcal{C} &\longmapsto \mathcal{C}_1 \\ \mathcal{C} &\longmapsto \mathcal{C}_2 = \{ X \xrightarrow{f} Y \xrightarrow{g} Z \} \\ \mathcal{C} &\longmapsto \mathcal{C}_3 = \{ \longrightarrow \longrightarrow \longrightarrow \} \\ &\vdots \\ \mathcal{C} &\longmapsto \mathcal{C}_n = \{ \longrightarrow \longrightarrow \longrightarrow \cdots \longrightarrow \} \end{aligned}$$

$$\begin{aligned} Top^{op} &\longrightarrow Set \\ (X, \tau) &\longmapsto \tau \subseteq 2^X \end{aligned}$$

$\exists$ topological space S w/ the property

$$\left\{ \begin{matrix} \text{continuous function} \\ X \longrightarrow S \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{open} \\ \text{subsets} \\ \text{of } X \end{matrix} \right\},$$

$S = \{a, b\}$

$\{\emptyset, \{a\}, \{a, b\}\}$

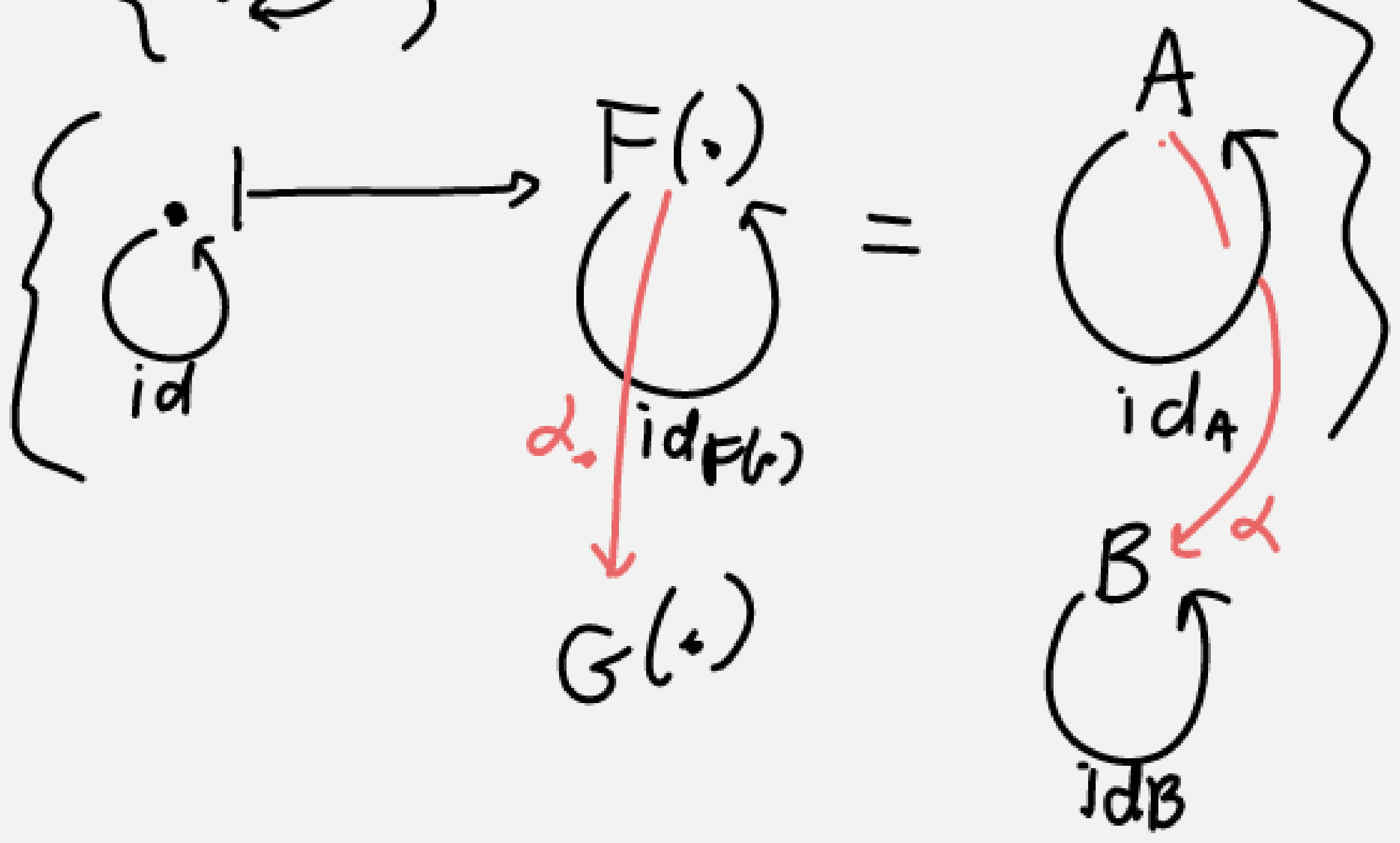
There exists a category $[\mathcal{C}^{o}, \mathrm{Set}]$ of presheaves (= contravariant functors from $\mathcal{C}$ to Set) and natural transformations

(This is what made the previous def'n sensible) bc natural isos are exactly isomorphisms $[\mathcal{C}^{o}, \mathrm{Set}]$

Examples:

$\mathcal{C} = \emptyset \rightsquigarrow [\mathcal{C}^{o}, \mathrm{Set}]$ is $\{\bullet \circlearrowleft\}$

$\mathcal{C} = \{\bullet \circlearrowleft id\} \rightsquigarrow [\mathcal{C}^{o}, \mathrm{Set}]$ is



$= \mathrm{Set}$

$\mathcal{C} = \{0, 1\}$

$\mathcal{C}^{op} \xrightarrow{F} Set$ sends $0 \longmapsto$ a set $F(0)$

$1 \longmapsto$ a set $F(1)$

(& identities to identities)

$[\mathcal{C}^{op}, Set]$ is the category $Set \times Set$

$F \longmapsto (F0, F1)$

$F_{AB} \left| \begin{array}{l} 0 \mapsto A \\ 1 \mapsto B \end{array} \right. \longleftarrow\!\shortmid (A, B)$

More generally regarding A^{set} as a discrete category

$[A^{op}, Set] \equiv Set^A$

$A^{op} = A$

$\prod_{a \in A} Set = Set^A$

Rmk Observe that this argument works more generally if $\mathcal{C}$ = discrete on a set A, $\{\text{functions } \mathcal{C} \xrightarrow{F} \mathbb{D}\} \equiv \{\text{functions } A \xrightarrow{F_0} \mathbb{D}_0\}$

objects of $\mathbb{D}$

$\mathcal{C} = \{ 0 \overset{s}{\underset{t}{\rightrightarrows}} 1 \}$

$\mathcal{C}^{op} \xrightarrow{G} Set$

$G(0) = \text{Vertices}$

$Gs\ \sigma \uparrow \ \uparrow Gt\ \tau$

$G(1) = \text{Edges}$

directed graph

$dGph = [\{0 \rightrightarrows 1\}^{op}, Set]$

$hom(-,0) : \{0 \rightrightarrows 1\}^{op} \to Set$

$hom(-,1) :$ " "

What is a representable digraph?

$hom(-,0)(0) = hom(0,0) = \{id_0\}$

$\underbrace{(1) = hom(1,0) = \cancel{\{\ \}}\ \emptyset}_{\Gamma_0}$

$\{ \underset{id_0}{\bullet} \}$

$hom(-,1)(0) = hom(0,1) = \{s,t\}$

$\underbrace{hom(-,1)(1) = hom(1,1) = \{id_1\}}_{\Gamma_1}$

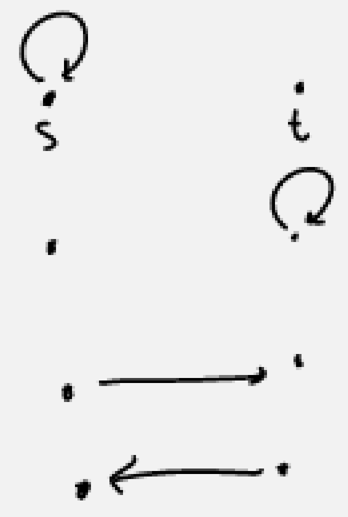
$= \Gamma_1(0) = \{ s \ , \ t \}$

$hom(s,1) = \sigma \uparrow \ \uparrow \tau = hom(t,1)$

$= \Gamma_1(1) = \{ \underset{x}{id_1} \}$

$\sigma(x) = x \circ s = id \circ s = s$

$\tau(x) = x \circ t = id \circ t = t$



(Yoneda lemma: $G \in dGph$

$G : \{0 \rightrightarrows 1\}^{op} \longrightarrow Set$

$G \iff \begin{matrix} \alpha_0 : \Gamma_0 \Rightarrow G \\ \alpha_1 : \Gamma_1 \Rightarrow G \end{matrix}$

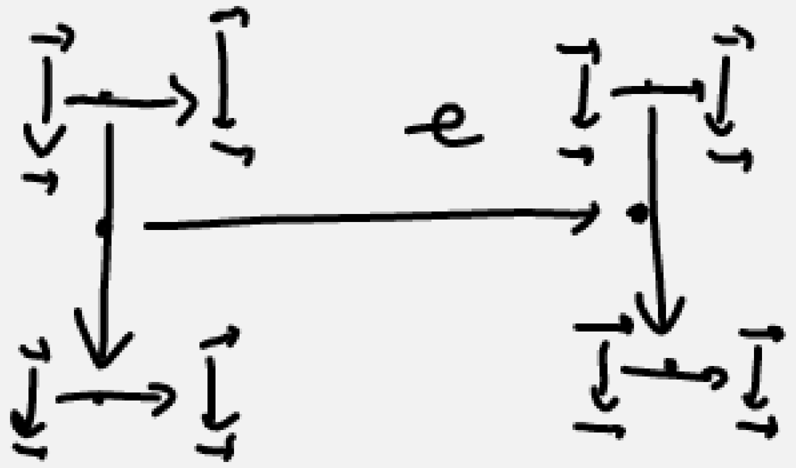
$\alpha_0 : \{\bullet\} \longrightarrow G$

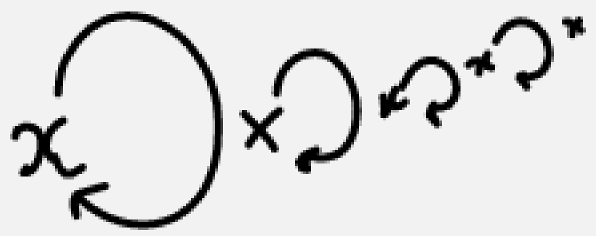
$\alpha_1 : \{\bullet \to \bullet\} \longrightarrow G$

$$F(2) = F(s,t) = \{[\,],\ \underbrace{[s],\ [t]},\ [st],\ [ts],\ [sst],\ [sts],\ [stt]\ldots\}$$

$$\underbrace{[ss]\quad [tt]}$$

free monoid on $\{s,t\}$





$$X : F(2) \longrightarrow Set$$

$$\longmapsto \quad X \underset{t}{\overset{s}{\rightrightarrows}} X$$

$$X_0 \subset X$$

$$X_0 = \{x \in X \mid s(x) = x = t(x)\}$$