

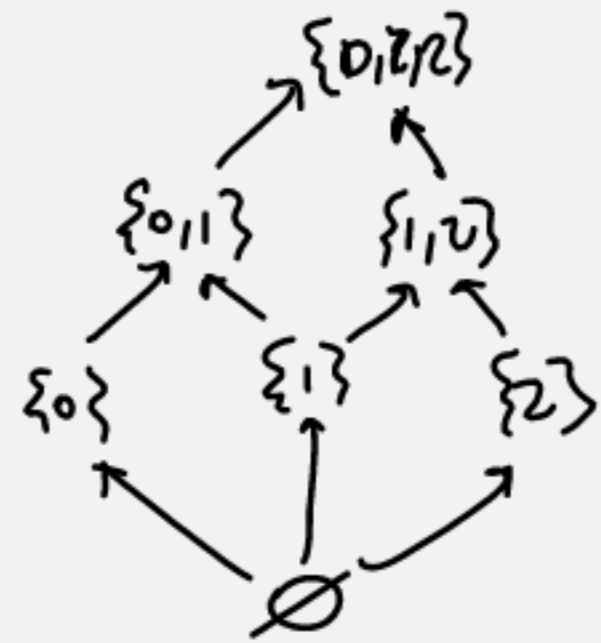
A preorder is a functor  $\mathcal{C}^{op} \rightarrow \text{Set}$

$$F : \mathcal{C}^{op} \rightarrow \text{Set}$$

$$C \longmapsto FC$$

$$C' \longmapsto FC'$$

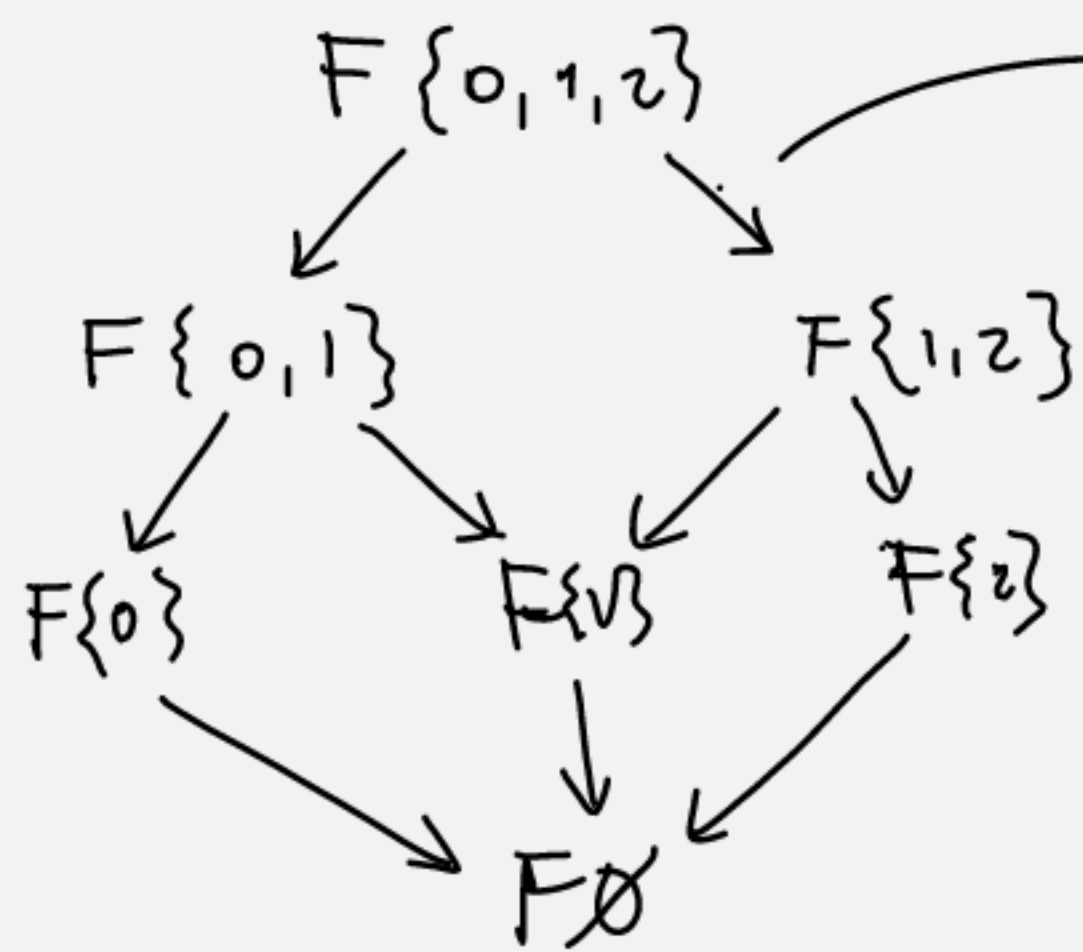
$$u \downarrow \quad \uparrow F_u$$



$$F(v \circ u) = F_u \circ F_v$$

$$F(A \xrightarrow{u} B \xrightarrow{v} C) = FA \xleftarrow{F_u} FB \xleftarrow{F_v} FC$$

Take  $\mathcal{C} =$  a poset, e.g.  $\text{subsets of } \{0, 1, 2\} = \{ \emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\} \}$



think of these maps as "restrictions" of a function defined on  $\{0, 1, 2\}$  to  $\{1, 2\}$

In fact this is exactly what happens

when  $F(-) = \text{hom}(-, X)$

$F(\{1, 2\} \subseteq \{0, 1, 2\})$  is the restriction map taking  $u : \{0, 1, 2\} \rightarrow X$  to  $u|_{\{1, 2\}} : \{1, 2\} \rightarrow X$

Any functor  $F: \mathcal{C} \rightarrow \text{Set}$   
 defines a family of sets

$$\{ FC \mid C \in \mathcal{C}_0 \}$$

The disjoint union of all  $FC$

$\sum_C FC$  is an important invariant of  $F$ .

Definition  $F: \mathcal{C} \rightarrow \text{Set}$   
 the category of elements of  $F$

$\text{El}(F)$  has

$$\text{objects} = \sum_C FC$$

(more concretely: pairs  $(C, x \in FC)$ ).

$$\text{Morphisms } \begin{matrix} (C, x) \\ x \in FC \end{matrix} \xrightarrow{u} \begin{matrix} (C', y) \\ y \in FC' \end{matrix}$$

are  $u: C \rightarrow C'$  such that

$$\begin{array}{ccc} \text{the function } F(u): FC' & \longrightarrow & FC \\ \text{maps } y & \text{to } & x \\ y & \longmapsto & x \end{array}$$

Identity of  $(C, x)$  is  $C \xrightarrow{\text{id}} C$  in  $\mathcal{C}$ .

$$\begin{array}{ccc} (C, x) & \longrightarrow & (C, x) \\ F(\text{id}_C) = \text{id}_{FC} & & (x) = x \end{array}$$

$$\begin{array}{ccccc} (C, x) & \xrightarrow{u} & (C', y) & \xrightarrow{v} & (C'', z) \\ \uparrow \text{FC} & & \uparrow \text{FC}' & & \uparrow \text{FC}'' \end{array}$$

$$\begin{array}{c} F(vu)(z) \\ Fu(Fv(z)) \\ Fu(y) \end{array}$$

Composition is done in  $\mathcal{C}$ ,

$$\begin{array}{ccccc} (C) & \xrightarrow{u} & (C') & \xrightarrow{v} & (C'') \\ (x) & & (y) & & (z) \end{array}$$

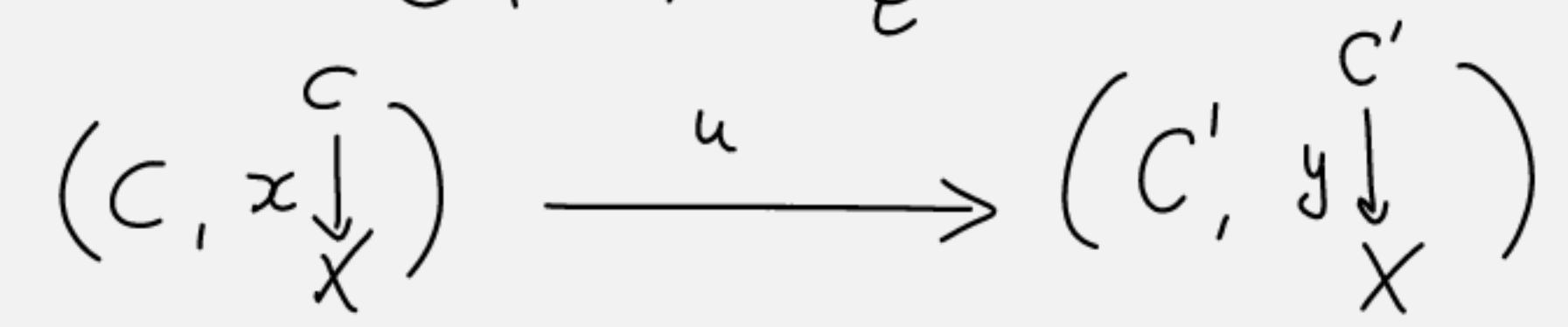
they are morphisms in  $\text{El}(F)$

$$\begin{array}{l} Fu(y) = x \\ Fv(z) = y \end{array} \implies \begin{array}{l} Fu(y) = Fu(Fv(z)) \\ \overset{x}{=} = F(vou)(z) \end{array}$$

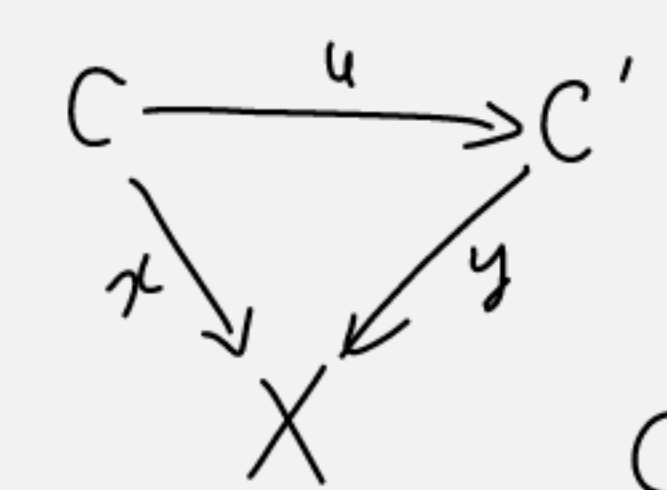
$\mathcal{E}\ell(F)$  when  $F$  is  $\text{hom}(-, X)$ ?

$\mathcal{E}\ell(\text{hom}(-, X))$  are pairs  $(C, x)$   
 $x \in \text{hom}(C, X)$   
 $C \xrightarrow{x} X$

$\text{hom}_{\mathcal{E}}(-, X) \equiv \mathcal{E}^{\text{op}} \rightarrow \text{Set}$   
 $C \mapsto \text{hom}_{\mathcal{E}}(C, X)$



is an arrow  $u: C \rightarrow C'$   
 $\text{hom}(u, X)(y) = x$

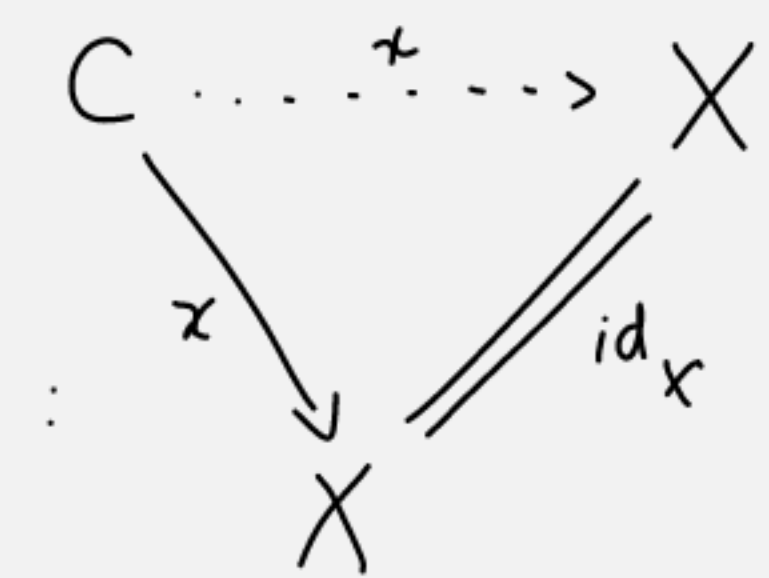


$\mathcal{E}\ell(\text{hom}(-, X)) = \mathcal{E}/X$

The category  $\mathcal{E}/X$  has a terminal object

(A category a t. obj in  $\mathcal{A}$   
 is  $T \in \mathcal{A}_0$  such that  
 for every other  $A \in \mathcal{A}_0$   
 $\exists!$  arrow  $A \rightarrow T$ )

If  $\mathcal{A} = \mathcal{E}/X$   
 $\text{id}_X$  is a terminal object:



What do I know about  $\mathcal{E}l(F)$ , if it has a terminal object?

Thm:  $F$  is representable ( $F \cong \text{hom}(-, X_F)$ ) iff  $\mathcal{E}l(F)$

If  $F$  is representable

$$FA \cong \text{hom}(A, X_F)$$

$$F(X_F) \cong \text{hom}(X_F, X_F)$$

$$\cong \sum_{\mathcal{F}}$$

$$F(-) = H^n(-, \mathbb{Z})$$

$$X_F = K(\mathbb{Z}, n)$$

$$H^n(K(\mathbb{Z}, n), \mathbb{Z}) \cong \sum_n$$

$$H^n(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)]$$

:

If  $F \cong \text{hom}(-, X)$   $\mathcal{L}(F) \cong \mathcal{L}/X$  which has  $\text{id}_X$  as terminal obj-

Now we have to prove the converse

Assume  $\mathcal{L}(F)$  has a terminal  $(T, t \in F(T))$

$$(C, x) \xrightarrow{\exists! h} (T, t)$$

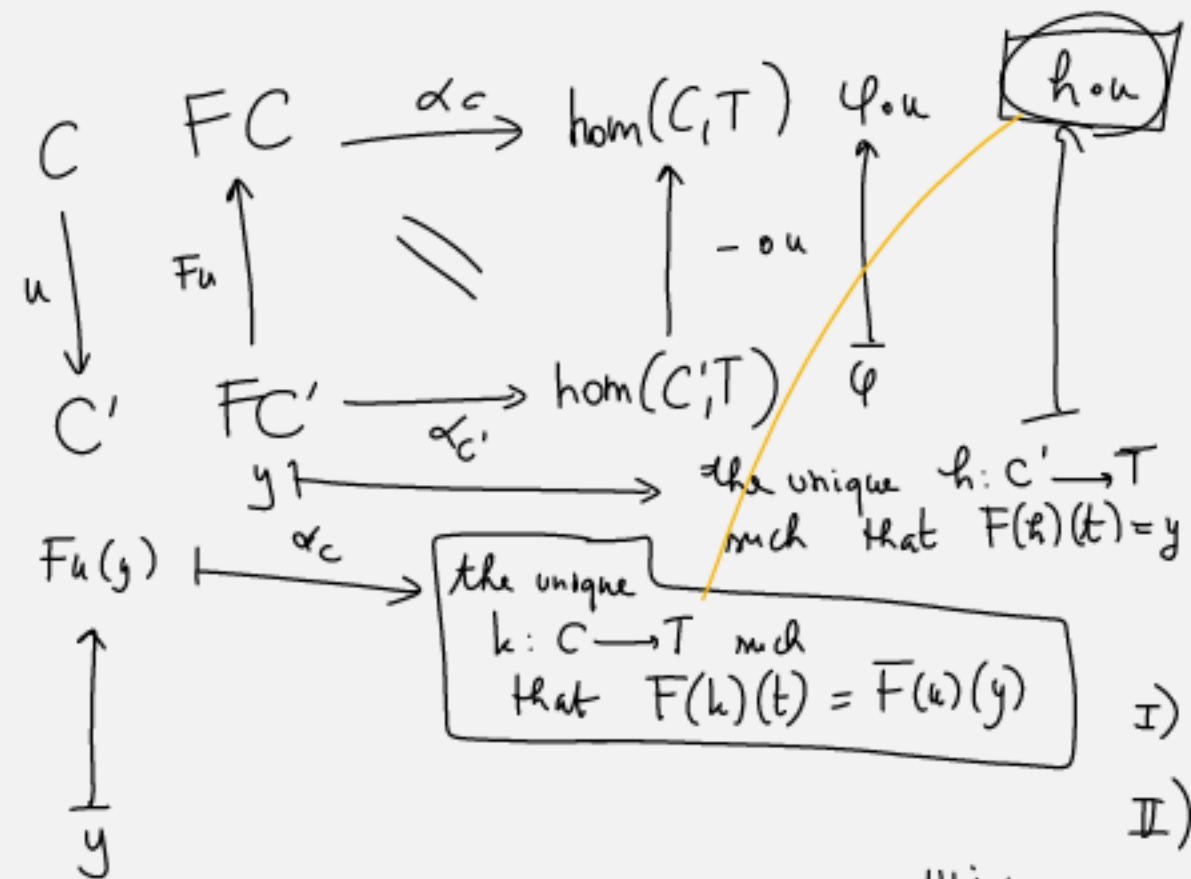
$\underbrace{\quad}_{FC}$

$\exists! h_{(C,x)}: C \rightarrow T$  with the property that  $F(h): F(T) \rightarrow F(C)$   
 $t \longmapsto x$   
 depends on both  $(C, x)$

This is to say that  $\alpha_C: FC \xrightarrow{\sim} \text{hom}(C, T)$  is a bijection  
 $x \longmapsto h_{(C,x)}$

Now check that  $\alpha_C$  defines the components of a natural transformation

$$\alpha_c: FC \xrightarrow{\sim} \text{hom}(C, T)$$



What if I knew that  
 I)  $F(h \circ u)(t) = F(u)(y)$   
 II)  $F(k)(t) = F(u)(y)$

uniqueness  $\Rightarrow k = h \circ u$

$$F(h \circ u)(t) = Fu(Fh(t)) = Fu(y)$$

Now I know that for every  $C$ ,  $FC \xrightarrow{\alpha_c} \text{hom}(C, T)$  are in (natural) bijection.

So in particular  $FT \cong \text{hom}(T, T)$

$\xi_F \xleftarrow{\sim} id_T$  is the unique point in  $FT$  such that  $\alpha_T(\xi_F) = id_T$

$\xi_F$  is called a universal element for  $F$

$$FC \xrightarrow{\alpha_c} \text{hom}(C, T)$$

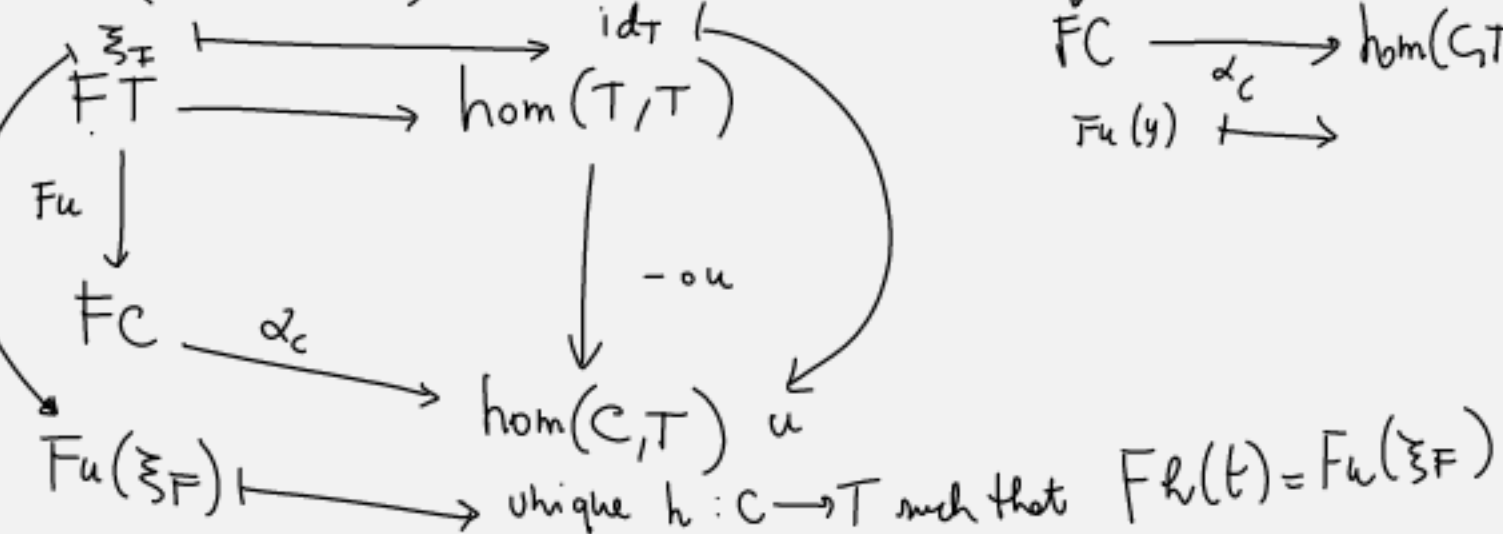
$$F(C \xrightarrow{u} C') = FC' \xrightarrow{Fu} FC$$

$$FC' \xrightarrow{\alpha_{c'}} \text{hom}(C', T)$$

$$Fu \downarrow \quad \downarrow - \circ u$$

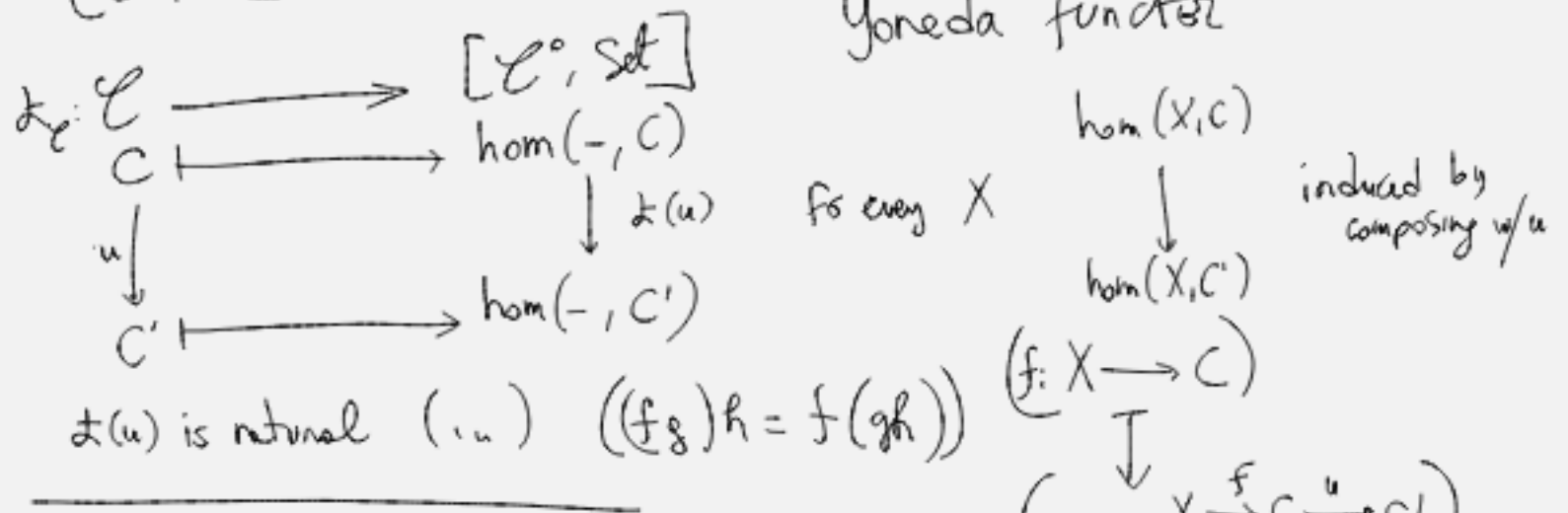
$$FC \xrightarrow{\alpha_c} \text{hom}(C, T)$$

$$Fu(y) \xrightarrow{\quad} \quad$$



for every  $u: C \rightarrow T$

$[\mathcal{C}^0, \text{Set}]$  is a category "containing"  $\mathcal{C}$ :



$\mathcal{Y}(u)$  is natural (i.e.)  $((f \circ g)h = f(gh))$

$$\begin{array}{ccc}
 (f: X \rightarrow \mathcal{C}) & & \\
 \downarrow & & \\
 (u \circ f: X \xrightarrow{f} \mathcal{C} \xrightarrow{u} \mathcal{C}') & & 
 \end{array}$$

$[\mathcal{C}^0, \text{Set}]$  contains  $\mathcal{C}$  in the sense

$\mathcal{Y}: \mathcal{C} \hookrightarrow [\mathcal{C}^0, \text{Set}]$  is an "embedding" in the following sense.

Def:  $F: \mathcal{C} \rightarrow \mathcal{D}$  functor b/w cats

- is
  - 1) FAITHFUL if each function  $\text{hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{hom}_{\mathcal{D}}(FX, FY)$  is injective
  - 2) FULL if each  $\text{hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{hom}_{\mathcal{D}}(FX, FY)$  is surjective
  - 3) FULLY FAITHFUL if it's both

- 1)  $Ff = Fg \implies f = g$
- 2) given  $FX \xrightarrow{h} FY$  is of the form  $F(u)$  for  $u: X \rightarrow Y$
- 3) (= 1+2) means that as above is unique every  $h: FX \rightarrow FY$  is of the form  $Fu$  for some a unique  $u: X \rightarrow Y$

Thm  $\mathcal{Y}: \mathcal{C} \longrightarrow [\mathcal{C}^0, \text{Set}]$

is fully faithful

$\mathcal{C} \cong$  the subcategory of  $[\mathcal{C}^0, \text{Set}]$  generated by the representable functors.

$\text{hom}_{\mathcal{C}}(X, Y)$  is in bijection with the set of natural transformations  $\text{hom}(-, X) \implies \text{hom}(-, Y)$