A presheaf is a functor $\mathcal{C}^{op} \longrightarrow Set$

$$\begin{tikzcd} F : \mathcal{C}^{op} \arrow[r] & Set \\ C \arrow[r, mapsto] \arrow[d, "u"'] & FC \\ C' \arrow[r, mapsto] & FC' \arrow[u, "Fu"'] \end{tikzcd}$$

$$F(v \circ u) = Fu \circ Fv$$

$$F(A \xrightarrow{u} B \xrightarrow{v} C) = FA \xleftarrow{Fu} FB \xleftarrow{Fv} FC$$

$$\begin{tikzcd} & & \{0,1,2\} & & \\ & \{0,1\} \arrow[ur] & & \{1,2\} \arrow[ul] & \\ \{0\} \arrow[ur] & & \{1\} \arrow[ul] \arrow[ur] & & \{2\} \arrow[ul] \\ & & \varnothing \arrow[ull] \arrow[u] \arrow[urr] & & \end{tikzcd}$$

Take $\mathcal{C}$ = a poset, e.g. subsets of $\{0,1,2\} = \{\varnothing, \{0\}, \{1\}, \{2\}, \{1,2\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$

$$\begin{tikzcd} & & F\{0,1,2\} \arrow[dl] \arrow[dr] & & \\ & F\{0,1\} \arrow[dl] \arrow[dr] & & F\{1,2\} \arrow[dl] \arrow[dr] & \\ F\{0\} \arrow[drr] & & F\{1\} \arrow[d] & & F\{2\} \arrow[dll] \\ & & F\varnothing & & \end{tikzcd}$$

think of these maps as "restrictions" of a function defined on $\{0,1,2\}$ to $\{1,2\}$

In fact this is exactly what happens when $F(-) = \hom(-, X)$

$F(\{1,2\} \subseteq \{0,1,2\})$ is the restriction map ~~of~~ taking $u : \{0,1,2\} \longrightarrow X$ to $u|_{\{1,2\}} : \{1,2\} \longrightarrow X$

Any functor $F : \mathcal{C}^{op} \longrightarrow Set$ defines a family of sets $\{ FC \mid C \in \mathcal{C}_0 \}$

The disjoint union of all FC, $\sum_C FC$ is an important invariant of F.

Definition $F : \mathcal{C}^{op} \longrightarrow Set$

the category of elements of F

$\mathcal{E}l(F)$ has

objects $= \sum_C FC$

(more concretely : pairs $(C, x \in FC)$.

Morphisms $\underset{x \in FC}{(C, x)} \xrightarrow{u} \underset{y \in FC'}{(C', y)}$

are $u : C \longrightarrow C'$ such that the function $F(u) : FC' \longrightarrow FC$ maps y to x. $y \longmapsto x$

Identity of (C, x) is $C \xrightarrow{id} C$ in $\mathcal{C}$.

$(C, x) \longrightarrow (C, x)$

$F(id_C) = id_{FC}(x) = x$

Composition is done in $\mathcal{C}$,

$\binom{C}{x} \xrightarrow{u} \binom{C'}{y} \xrightarrow{v} \binom{C''}{z}$

they are morphisms in $\mathcal{E}l(F)$

$Fu(y) = x$

$Fv(z) = y$

$\Longrightarrow$ $\underset{x}{Fu(y)} = Fu(Fv(z)) = F(v \circ u)(z)$

$\underset{\in FC}{(C, x)} \xrightarrow{u} \underset{\in FC'}{(C', y)} \xrightarrow{v} \underset{\in FC''}{(C'', z)}$

$F(vu)(z)$

$Fu(Fv(z))$

$Fu(y)$

x

$\mathcal{El}(F)$ when F is $\hom(-,X)$?

$\mathcal{El}(\hom(-,X))$ are pairs (C, x)
$x \in \hom(C, X)$

$C \xrightarrow{x} X$

$\hom_{\mathcal{C}}(-,\underline{\underline{X}}) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

$C \longmapsto \hom_{\mathcal{C}}(C, X)$

$$\left(C, \begin{array}{c} C \\ {\scriptstyle x}\downarrow \\ X \end{array}\right) \xrightarrow{\quad u \quad} \left(C', \begin{array}{c} C' \\ {\scriptstyle y}\downarrow \\ X \end{array}\right)$$

is an arrow $u : C \longrightarrow C'$

$\underline{\hom(u, X)(y)} = x$

$$\begin{tikzcd} C \arrow[rr, "u"] \arrow[dr, "x"'] & & C' \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

$$\mathcal{El}(\hom(-,X)) = \mathcal{C}/X$$

The category $\mathcal{C}/X$ has a <u>terminal object</u>

($\mathcal{A}$ category a t. obj in $\mathcal{A}$ is $T \in \mathcal{A}_0$ such that for every other $A \in \mathcal{A}_0$ $\exists!$ arrow $A \longrightarrow T$

If $\mathcal{A} = \mathcal{C}/X$

id_X is a terminal object :

$$\begin{tikzcd} C \arrow[rr, dotted, "x"] \arrow[dr, "x"'] & & X \arrow[dl, equal, "id_X"] \\ & X & \end{tikzcd}$$

What do I know about $\mathcal{El}(F)$, if it has a terminal object?

<u>Thm</u>: F is representable $\left(F \cong \hom(-, X_F) \right)$ iff $\mathcal{El}(F)$

If F is representable

$FA \cong \hom(A, X_F)$

$F(X_F) \cong \hom(X_F, X_F)$

$\ni$

ξ_F

$F(-) = H^n(-, \mathbb{Z})$

$X_F = K(\mathbb{Z}, n)$

$H^n(K(\mathbb{Z}, n), \mathbb{Z}) \ni \xi_n$

$$H^n(X, \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)]$$

$\vdots$

If $F \cong \hom(-, X)$ $\quad \mathcal{E}l(F) \cong \mathcal{C}/X$ which has id_X as terminal obj.

Now we have to prove the converse

Assume $\mathcal{E}l(F)$ has a terminal

$$(T, \; t \in F(T))$$

$$\underset{\substack{\cap \\ FC}}{(C, x)} \xrightarrow{\exists ! \, h} (T, t)$$

$\exists ! \; h_{(C,x)} : C \longrightarrow T$ with the property that $F(h) : F(T) \longrightarrow F(C)$

$$t \longmapsto x$$

depends on both (C, x)

This is to say that $\alpha_C : FC \xrightarrow{\sim} \hom(C, T)$ is a bijection

$$x \longmapsto h_{(C,x)}$$

Now check that defines the components of a nat trns

$$\alpha_C : FC \longrightarrow \hom(C,T)$$
$$x \longmapsto h(C,x)$$

$$\begin{tikzcd}
C \arrow[d, "u"'] & FC \arrow[r, "\alpha_C"] & \hom(C,T) & \varphi \circ u & \boxed{h \circ u} \\
C' & FC' \arrow[u, "Fu"] \arrow[r, "\alpha_{C'}"'] \arrow[ur, phantom, "="] & \hom(C',T) \arrow[u, "- \circ u"'] & \varphi \arrow[u, mapsto] & {} \arrow[u, mapsto]
\end{tikzcd}$$

$y \longmapsto$ the unique $h : C' \longrightarrow T$ such that $F(h)(t) = y$

$Fu(y) \overset{\alpha_C}{\longmapsto}$ $\boxed{\text{the unique } k : C \longrightarrow T \text{ such that } F(k)(t) = F(u)(y)}$

$y \longmapsto Fu(y)$

What if I knew that

I) $\boxed{F(h \circ u)(t) = F(u)(y)}$

II) $F(k)(t) = F(u)(y)$

Uniqueness $\Rightarrow k = h \circ u$

$$F(h \circ u)(t) = Fu\left(\underbrace{Fh(t)}\right)$$
$$= Fu(y)$$

Now I know that for every C, $FC \overset{\alpha_C}{\cong} \hom(C,T)$ are in (natural) bijection.

So in particular $FT \cong \hom(T,T)$

$\xi_F \longleftarrow\!\!\shortmid \mathrm{id}_T$

is the unique point in $F(T)$ such that $\alpha_T(\xi_F) = \mathrm{id}_T$

ξ_F is called a universal element for F

$$FC \xrightarrow[\sim]{\alpha_C} \hom(C,T)$$

$$F\left(C \xrightarrow{u} C'\right) = FC' \xrightarrow{Fu} FC$$

$$\begin{tikzcd}
y & FC' \arrow[r, "\alpha_{C'}"] \arrow[d, "Fu"'] & \hom(C',T) \arrow[d, "- \circ u"] \\
 & FC \arrow[r, "\alpha_C"] & \hom(C,T)
\end{tikzcd}$$

$Fu(y) \longmapsto$

$$\begin{tikzcd}
\xi_F \arrow[r, mapsto] \arrow[dd, bend right] & \mathrm{id}_T \arrow[dd, bend left] \\
FT \arrow[r] \arrow[d, "Fu"'] & \hom(T,T) \arrow[d, "- \circ u"] \\
FC \arrow[r, "\alpha_C"] & \hom(C,T) & u
\end{tikzcd}$$

$Fu(\xi_F) \longmapsto$ unique $h : C \longrightarrow T$ such that $Fh(t) = Fu(\xi_F)$

for every

$u : C \longrightarrow T$

$[\mathcal{C}^\circ, \mathrm{Set}]$ is a category "containing" $\mathcal{C}$:

Yoneda functor

$$\begin{array}{ll} \mathcal{Y}_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^\circ, \mathrm{Set}] \\ C \longmapsto \hom(-, C) \\ u \downarrow \qquad\quad \downarrow \mathcal{Y}(u) \\ C' \longmapsto \hom(-, C') \end{array}$$

for every X

$$\begin{array}{c} \hom(X, C) \\ \downarrow \\ \hom(X, C') \end{array}$$

induced by composing w/ u

$$\begin{array}{c} (f : X \longrightarrow C) \\ \downarrow \\ (u \circ f : X \xrightarrow{f} C \xrightarrow{u} C') \end{array}$$

$\mathcal{Y}(u)$ is natural $(\ldots)$ $((fg)h = f(gh))$

$[\mathcal{C}^\circ, \mathrm{Set}]$ contains $\mathcal{C}$ in the sense

$\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \hookrightarrow [\mathcal{C}^\circ, \mathrm{Set}]$ is an "embedding"

in the following sense.

Def : $F : \mathcal{C} \longrightarrow \mathbb{D}$ functor btwn cats

is 1) FAITHFUL if each function

$$\hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathbb{D}}(FX, FY) \text{ injective}$$

2) FULL if each

$$\hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathbb{D}}(FX, FY) \text{ surjective}$$

3) FULLY FAITHFUL if it's both

1) $Ff = Fg \implies f = g$

2) given $FX \xrightarrow{h} FY$ is of the form $F(u)$ for $u : X \longrightarrow Y$

3) $(= \underline{1+2})$ means h as above is unique

every $h : FX \longrightarrow FY$ is of the form Fu for ~~som~~ a <u>unique</u> $u : X \to Y$

<u>Thm</u> $\mathcal{Y}_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^\circ, \mathrm{Set}]$

is fully faithful

$\mathcal{C} \cong$ the subcategory of $[\mathcal{C}^\circ, \mathrm{Set}]$

generated by the representable functors.

$\underline{\hom_{\mathcal{C}}(X, Y)}$ is in bijection with the set

of natural transformations

$$\hom(-, X) \Longrightarrow \hom(-, Y)$$