$$\mathcal{C} \xrightarrow{よ} [\mathcal{C}^\circ, \mathrm{Set}] \qquad X \longmapsto \mathcal{C}(-,X) \ (= \hom(-,X))$$

$$\mathcal{C}^\circ \xrightarrow{\text{co}\,よ} [\mathcal{C}, \mathrm{Set}] \qquad A \longmapsto \mathcal{C}(A,-) \ (= \hom(A,-))$$

$$\mathcal{C}^\circ \times \mathcal{C} \xrightarrow{\hom} \mathrm{Set} \qquad \text{hom functor}$$

Yoneda & coYoneda arise from "saturating" hom in one or the other variable

$$\begin{array}{ccc} (A,X) & \longmapsto & \hom(A,X) \\ {\scriptstyle u}\uparrow \ \downarrow{\scriptstyle f} & & \downarrow \\ (B,Y) & & \hom(B,Y) \end{array} \qquad A \xrightarrow{g} X \qquad \begin{pmatrix} B & & Y \\ {\scriptstyle u}\downarrow & & \uparrow{\scriptstyle f} \\ A & \xrightarrow[g]{} & X \end{pmatrix} \qquad \begin{array}{c} g \\ \downarrow \\ f \circ g \circ u \end{array}$$

$よ$, co$よ$ arise from an op'n on categories which is called

CURRYING

$$[\mathcal{A} \times \mathcal{B}, \mathcal{C}] \cong [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

$$[\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$$

To every $F : \mathcal{C}^{\circ} \longrightarrow Set$ associates the category of elements

$\mathcal{E}lts(F)$

$$(A,\ a \in FA) \xrightarrow{h} (B,\ b \in FB)$$

is $h : A \longrightarrow B$ such that $Fh(b) = a$.

F is naturally isom. to $\underline{\hom(-, X_F)}$ $\iff$

$\mathcal{E}lts(F)$ has a terminal object

$$FA \cong \hom(A, X_F) \qquad (A, a) \xrightarrow{h_{(A,a)}!} (X_F,\ \xi_F \in FX_F)$$

$$a \longleftrightarrow h_{(A,a)}$$

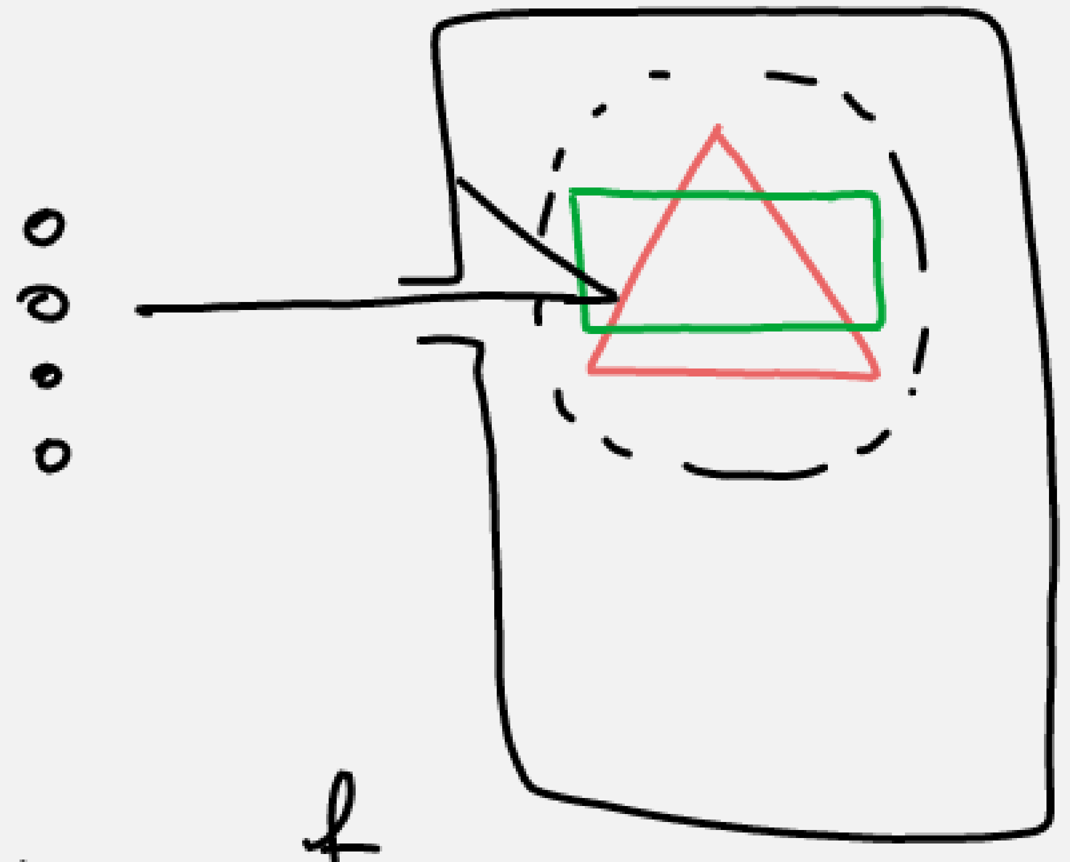
$\xi_F \in F X_F$

$$F(X_F) \simeq \hom(X_F, X_F)$$

arrows $X_F \to X_F$

$$\xi_F \leftarrow \boxed{id_{X_F}}$$

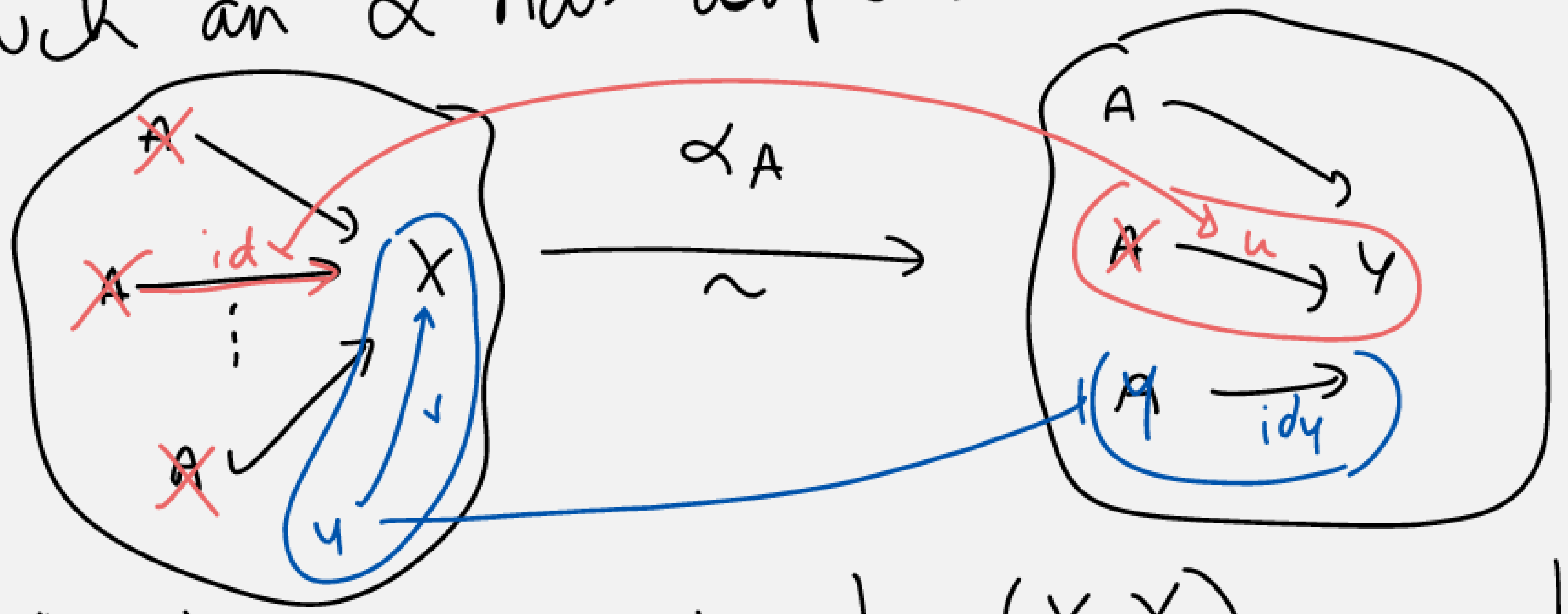
UNIVERSAL ELEMENT for F.



$$A_1 \xrightarrow{f} X, \quad A_2 \to X, \quad A_3 \to X$$

To decide whether $X, Y \in \mathcal{C}_0$ are isomorphic.
it is enough (necessary & sufficient) to
find an isomorphism between the
representable functors $\hom(-,X) \xrightarrow[\sim]{\alpha} \hom(-,Y)$

Such an α has components



$\forall A.$

If $\underline{A = X}$ $\qquad \alpha_X : \underline{\hom(X,X)} \longrightarrow \hom(X,Y)$

$$id \longmapsto u = \alpha_X(id_X)$$

similarly $\qquad \alpha_A^{-1} : \hom(A,Y) \longrightarrow \hom(A,X)$

$A = Y$ $\qquad \alpha_Y^{-1} : \hom(Y,Y) \longrightarrow \hom(Y,X)$

$$id_Y \longmapsto v = \alpha_Y^{-1}(id_Y)$$

YL I) $\alpha : \hom(-,X) \Longrightarrow F$

Natural trns of type are uniquely determined by the image of id_X

$\alpha_A : \hom(A,X) \longrightarrow FA$ acts in a way that is completely determined by $\alpha_X(id_X) \in FX$

(This means that given α, β $\begin{matrix}\alpha : \\ \beta :\end{matrix} \hom(-,X) \Rightarrow F$ $\quad \alpha = \beta \; (\alpha_A = \beta_A \;\; \forall A)$

IFF $\underline{\alpha_X(id_X)} = \underline{\beta_X(id_X)}$)

YL II) Every element $\underline{\xi \in FX}$ extends to a natural transformation

$$\alpha : \hom(-,X) \Rightarrow F$$

such that $\alpha_X(id_X) = \xi \in FX$

YL I, YL II prove that

$$\mathrm{Nat}(\hom(-,X),F) \xrightarrow{\mathrm{I}} FX$$
$$\{\alpha_A : \hom(A,X) \to FA\} \longmapsto \alpha_X(id_X)$$

is a bijection (YL I injective) / (YL II surjective)

Yoneda Lemma

$\left\{ \begin{matrix} \text{natural trns} \\ \hom(-,X) \Rightarrow F \end{matrix} \right\}$ in bijection with FX

In particular (as a corollary)

$F = \hom(-,Y)$

$\mathrm{Nat}(\hom(-,X), \hom(-,Y)) \cong \hom(-,Y)(X)$

$\hom(X,Y)$

$$\mathrm{Nat}(\hom(-,X), \hom(-,Y)) = \hom_{[\mathcal{C}^\circ, Set]}(\text{よ}X, \text{よ}Y)$$

$$\hom_{\mathcal{C}}(X,Y) \cong \hom_{[\mathcal{C}^\circ, Set]}(\text{よ}X, \text{よ}Y)$$

which is the def'n of fully faithful functor

YL I

Given $\alpha : \hom(-,X) \Rightarrow F$, for every $u : A \longrightarrow B$

$$
\begin{tikzcd}
\hom(B,X) \arrow[r, "\alpha_B"] \arrow[d, "-\circ u"'] & FB \arrow[d, "Fu"] \\
\hom(A,X) \arrow[r, "\alpha_A"'] & FA
\end{tikzcd}
\qquad
\begin{tikzcd}
B \xrightarrow{f} X \arrow[r, mapsto] \arrow[d, mapsto, "-\circ u"'] & \alpha_B(f) \arrow[d, mapsto, "Fu"] \\
 & Fu(\alpha_B(f)) \\
A \xrightarrow{f\circ u} X \arrow[r, mapsto, "\alpha_A"'] & \alpha_A(f\circ u)
\end{tikzcd}
$$

$\alpha_A(f\circ u) = Fu(\alpha_B(f))$ $\Longrightarrow$

$\forall\ u : A \longrightarrow B$

$\forall\ f : B \longrightarrow X$ $\qquad f = \mathrm{id}_X : X \longrightarrow X$

$\alpha_A(\mathrm{id}\circ u) = Fu(\underbrace{\alpha_X(\mathrm{id}_X)}_{\xi_F})$

$\forall u \quad \alpha_A(u) = Fu(\xi_F)$

$\alpha_A(-) =$ evaluation at ξ_F of Fu

The expression $\alpha_A(u) = Fu(\underbrace{\xi_F^{\alpha}}_{\alpha_X(\mathrm{id}_X)})$

$\beta_A(u)$ (for β of same type $\hom(-,X) \Rightarrow F$)

$= Fu(\underbrace{\xi_F^{\beta}}_{\beta_X(\mathrm{id}_X)})$

YL II)

Given $\xi \in FX$, have to define a n. transformation

$$\alpha : \hom(-, X) \Rightarrow F$$

$$\begin{aligned} \alpha_A : \hom(A, X) &\longrightarrow FA \\ (f : A \to X) &\longmapsto \end{aligned}$$

Cook an element of FA from
0 — $X \in \mathcal{C}_0$
1 — $F : \mathcal{C}^{op} \to Set$,
2 — $\xi : FX$
3 — $f : A \to X$

$$\begin{tikzcd} FX \arrow[r, "F(f)"] & FA \\ \xi \arrow[r, mapsto] & F(f)(\xi) \end{tikzcd}$$

$$\alpha_A^{(\xi)}(f) = F(f)(\xi)$$

btw by definition $\alpha_{\cancel{A}X}^{(\xi)}(id_X) = F(id_X)(\xi)$
$= id_{F(X)}(\xi)$
$= \xi$

$$\underline{\alpha_A^{(\xi)}} = \underline{\lambda f.\ F(f)(\xi)}$$

$\{\alpha_A\}$ is a nat. transformation:

$$
\begin{tikzcd}
A \arrow[d, "u"'] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
\hom(B, X) \arrow[r, "\alpha_B^{(\xi)}"] \arrow[d, "-\circ u"'] & FB \arrow[d, "Fu"] \\
\hom(A, X) \arrow[r, "\alpha_A^{(\xi)}"'] & FA
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(B \xrightarrow{f} X\right) \arrow[r, mapsto] \arrow[d, mapsto, "-\circ u"'] & F(f)(\xi) \arrow[d, mapsto] \\
 & Fu\left(F(f)(\xi)\right) \\
\left(A \xrightarrow{f\circ u} X\right) \arrow[r, mapsto] & F(f \circ u)(\xi)
\end{tikzcd}
$$

$$Fu\left(F(f)(\xi)\right) = (Fu \circ Ff)(\xi)$$

$$F(f \circ u)(\xi) = (Fu \circ Ff)(\xi)$$

$(P, \leq)$ partially ordered set

$P^\circ \longrightarrow Set$

$[P^\circ, Set]$

$P \longrightarrow [P^\circ, Set]$

$x \longmapsto \hom_P(-, x)$

$a \longmapsto \hom(a, x)$ — $\{a \leq x\}$ if $a \leq x$; $\emptyset$ otherwise

" initial segment " of x

YL says $\mathrm{Nat}(\hom(-,x), F) \cong F(x)$

$\alpha_a : \hom(a, x) \longrightarrow Fa$

$a \not\leq x$: $\alpha_a : \emptyset \longrightarrow Fa$

$a \leq x$: $\{*\} \longrightarrow Fa$

$\alpha : \hom(-, x) \longrightarrow \hom(-, y)$

$\alpha_a : \underline{\hom(a, x)} \longrightarrow \hom(a, y)$

if $a \leq x$ then $a \leq y$

$\alpha_x(x \leq x) \in \hom(x, y)$ saying that $x \leq y$

$\alpha_y^{-1}(y \leq y) \in \hom(y, x)$ " " $y \leq x$

$\alpha^{-1} : \hom(-, y) \longrightarrow \hom(-, x)$

is the converse

if $a \leq y$ then $a \leq x$

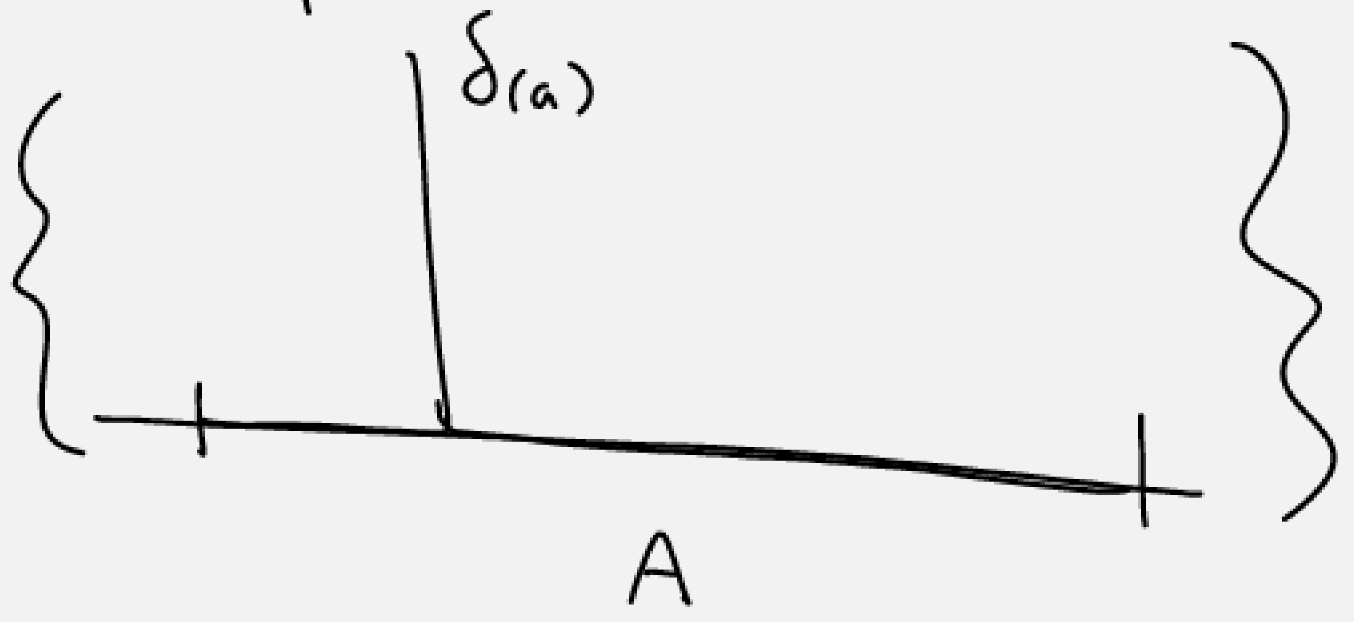
$\Longrightarrow x = y$

A set (discrete category)

$A^{op} \longrightarrow Set$ amounts to a family of sets $\{F_a \mid a \in A\}$

$$A \xrightarrow{よ_A} [A^{op}, Set]$$

$$a \longmapsto \hom_A(-, a) \quad \begin{cases} \varnothing & x \neq a \\ \{id_a\} & x = a \end{cases}$$



$\delta(a)$ for every a is a family of "Dirac" distribution

YL says that a generic "distribution $A^{op} \xrightarrow{F} Set$

is reconstructed from

$$\{\delta(a) \Longrightarrow F\} = F_a$$

$$\underbrace{\delta(a)(x)}_{} \longrightarrow F(x)$$

$\varnothing$ $a \neq x$

$$\{\{*\} \longrightarrow F(a)\} \equiv F(a)$$