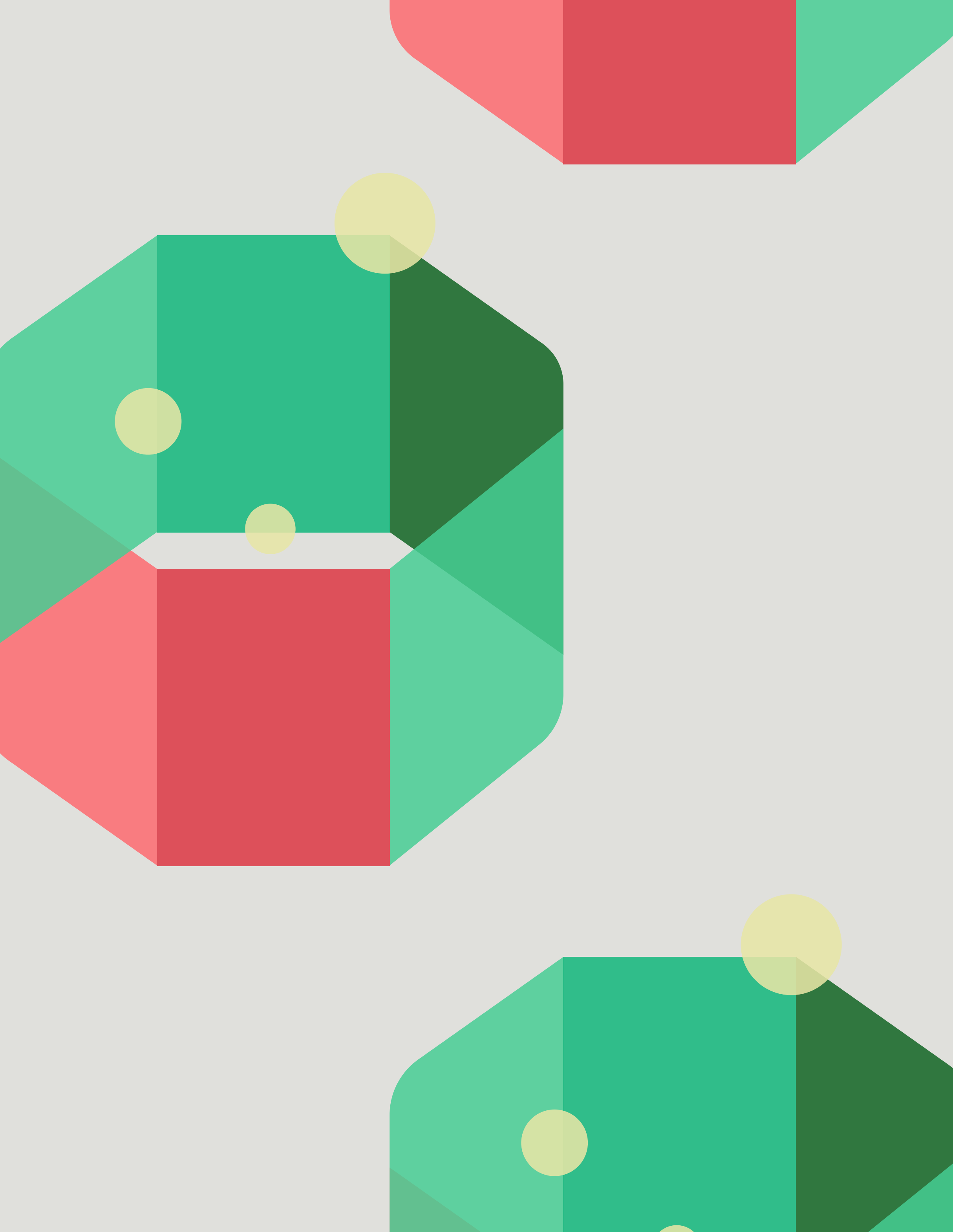


Yesterday = YL

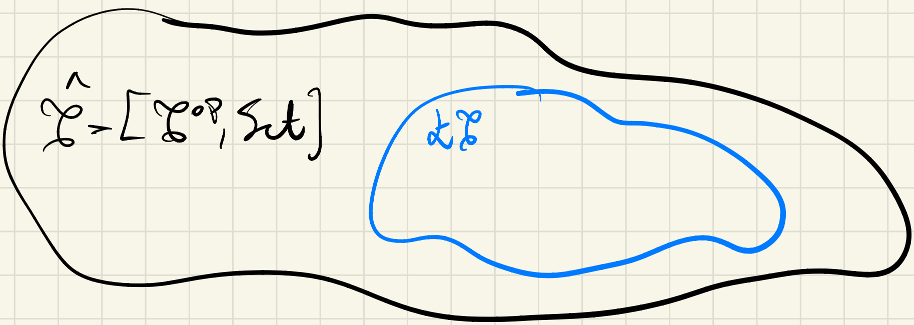
Today = instances of YL in categories we know / care about

YL ≈ Understand some functors well

actually YL = description of a subcategory of $[\mathcal{C}^{op}, Set]$

subcategory of representable $\simeq \hom(-,x) : \mathcal{C}^{op} \to Set$

subcat of repr is equivalent to $\mathcal{C}$



$$よ_{\mathcal{C}} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, Set] \quad \text{is fully faithfull}$$
$$x \longmapsto \hom(-,x)$$

$$\{ よ_{\mathcal{C}} X \Rightarrow よ_{\mathcal{C}} Y \} \cong \{ X \longrightarrow Y \}$$

If you want $X \cong Y$, you can instead:

i) take arbitrary $A \in \mathcal{C}$

ii) Prove $\hom(A, X) \cong \hom(A, Y)$ naturally bijective

ii) looks more complicated

but it's an iso of Sets

which may be easier to obtain

than iso of obj in $\mathcal{C}$.

eg $\text{Hom}(A, X) \cong \hom(A, E_1) \cong$

$\vdots$

$\hom(A, E_n) \cong$

$\hom(A, Y)$:)

Today, more examples

i) $Mat(\mathbb{R})$ • obj are $n \in \mathbb{N}$

• $n \xrightarrow{\mathbb{R}} m \leftrightarrow n \times m$ matrices with entries in $\mathbb{R}$.

- What are repr functors?
- What are nts $よ_m \Rightarrow よ_n \cong m \to n$
- $\{よA \to F\} \cong FA$ det by id_A.

representables @ $k \in \mathbb{N}$

$h_k : Mat^{op} \longrightarrow Set$

$n \longmapsto Mat(n,k) = \{ \downarrow_k \overset{n}{\underset{n}{\longrightarrow}} \}$

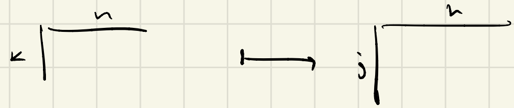
$h_k(n \xrightarrow{A} m) = Mat(m,k) \longrightarrow Mat(n,k)$

$\underset{m \xrightarrow{B} k}{B\ {}^{m}|_k} \longmapsto \underset{n \xrightarrow{A} m \xrightarrow{B} k}{B\ |_k \ {}^{n}|_m A} = \underset{k \times n}{B \cdot A}$

n.t

$\alpha : h_k \to h_j$ has components:

$$\alpha_n : \mathrm{Mat}(n,k) \xrightarrow{\alpha_n} \mathrm{Mat}(n,j)$$



a) $h_k \xrightarrow{\delta} h_{k-1}$ by "delete last row"

$$\begin{tikzcd} h_k(n) \arrow[r, "\delta_n"] \arrow[d, "h_k(A)"'] & h_{k-1}(n) \arrow[d] \\ h_k(m) \arrow[r, "\delta_m"] & h_{k-1}(m) \end{tikzcd}$$

B

$B \cdot A$

$\boxed{\delta_n B}$

$\boxed{\delta_n B} \cdot \boxed{A}$ chop then mult

$=^*$:)

$\boxed{\delta_m (B \cdot A)}$ mult then chop

*proof by "trust Fosco"

b) $\gamma : h_k \to h_{k-1}$

$$\gamma_n : \mathrm{Mat}(n,k) \longrightarrow \mathrm{Mat}(n,k+1)$$

not natural trans

$h_k \Rightarrow h_{k-1}$

How do we tell apart the natural from the unnatural? YL

YL says $h_k \longrightarrow h_j$ natural ones arise by mult by some (fixed) matrix

YL says $\alpha : h_k \Rightarrow h_j$ then

$$A \in Mat(n,k) : \quad \alpha_n(A) = A \cdot \underbrace{\alpha_k(I_k)}_{\text{can be computed}}$$

$$\alpha_k : Mat(k,k) \longrightarrow Mat(k,j)$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \longmapsto \alpha_k(I_k)$$

eg δ = "delete last row"

$$\underbrace{\delta_n(A) =}_{\text{must be}} A \cdot \underbrace{\delta_k(I_k)}_{\text{compute } \cong I_{k-1}} = A \cdot I_{k-1}$$

eg π = swap $2 \leftrightarrow 4$ row

$$\pi_n(A) = A \cdot \underbrace{\pi_k(I_k)}_{} \longrightarrow \begin{pmatrix} 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1 \end{pmatrix}$$

What α? $\quad \alpha(AB) = A \cdot \alpha B \quad$ $\underline{\underline{*}}$

can only be $\quad \alpha(A) = A \cdot H_\alpha$

H_α fixed, depends on α.

then $\quad \alpha(AB) = (AB)H_\alpha \underset{\text{assoc!}}{=} A(BH_\alpha) = A(\alpha B)$

YL! These are the only α satisfying $*$.

$$[(\mathcal{C}/X)^{op}, \mathbf{Set}]$$

Given $\begin{tikzcd} A \arrow[d, "u"'] \\ X \end{tikzcd}$ $\rightsquigarrow$ $\left[\mathcal{C}/X(-,(u,A))\right](v,B) = \left\{ \begin{tikzcd} B \arrow[rr] \arrow[dr, "v"'] & & A \arrow[dl, "u"] \\ & X & \end{tikzcd} \right\}$

YL: The only way to build

$\alpha : よ(A,u) \Rightarrow よ(B,v)$ is from a

$$\begin{tikzcd} A \arrow[d, "u"] \\ X \end{tikzcd} \rightsquigarrow \begin{tikzcd} B \arrow[d, "v"] \\ X \end{tikzcd}$$

So, $\alpha_{C,w} : \mathcal{C}/X((C,w),(A,u)) \longrightarrow \mathcal{C}/X((C,w),(B,v))$

where to send this?

$$\begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "w"'] & & A \arrow[dl, "u"] \\ & {} & \end{tikzcd} \qquad \star \longmapsto \ ? \ \star$$

YL: Has to be comp by fixed el. Which el?

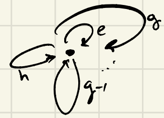
Look at $id_x \in \mathcal{C}/X((A,u),(A,u))$ and take its image

$$
\begin{tikzcd}
A \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & A \arrow[dl, "u"] \\
& X &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dr, "u"'] & & B \arrow[dl, "v"] \\
& X &
\end{tikzcd}
$$

$\cancel{A}$

$$
\begin{tikzcd}
\# \arrow[rr, "k"] \arrow[dr, "u"'] & & {} \arrow[dl, "v"] \\
& X &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
\# \arrow[r, "k"] \arrow[dr, "w"'] & A \arrow[r, "h"] \arrow[d, "u"] & B \arrow[dl, "v"] \\
& X &
\end{tikzcd}
$$

iii) G group then $BG^{op} \xrightarrow{r} Set$



To specify an r: $r(\bullet) = A$

$\forall g \in G = r(g) : A \xrightarrow{\cong} A$

g iso in $BG \Rightarrow r(g)$ iso in Set

satisfying functoriality: $\begin{cases} (r_g \circ r_h)(x) = r_{hg}(x) \\ r_{id_G}(x) = x \end{cases}$

same data as group hom $\quad (G, \cdot) \longrightarrow (Bij\,A, \circ)$

So, functor $BG^{op} \longrightarrow Set \equiv$ G-Action on A

$$\widehat{BG} \cong G\text{-}Sets$$

$$\widehat{BG} = [BG^{op}, Set]$$

THE "Representables are nice G-actions"

$\exists$ only one.

$BG(-, \bullet) : BG^{op} \longrightarrow Set$

$\bullet \longmapsto Hom(\bullet, \bullet) \cong G$

G is acting on itself

$$G \longrightarrow \mathrm{Bij}(G) \quad \in \mathrm{Grp\,Hom}$$

$$g \longmapsto (x \longmapsto x \cdot g) = m_g$$

hom functor = comp. of morphisms "BG"
= mult of elements "G"

m_g is bijective with inverse $y \mapsto y \cdot g^{-1}$.

YL: $\mu : G \longrightarrow \mathrm{Bij}\, G$ is bijective
$g \longmapsto m_g$

μ inj: assume $\mu g = \mu h \rightarrow m_g = m_h$

$\Leftrightarrow \forall x \quad m_g(x) = m_h(x)$

$\Leftrightarrow \quad x \cdot g = h \cdot x$

$\xrightarrow{x=e} \quad eg = eg$

$\Rightarrow \quad g = h \quad \checkmark$

Cayley Thm: $G \leq (\underset{(Set)}{\mathrm{Bij}}\ G_0 \to G_0, \circ)$