G directed graph, object of the category of graphs

a graph homomorphism is a pair of maps

$$f_0 : G_0 \longrightarrow H_0$$

$$f_1 : G_1 \longrightarrow H_1$$

preserving source & target

The objects of $\underline{Gph}$ of d. graphs can be presented as functors precisely a d. graph G consists of a functor from $\underline{uG}$ $\left\{ 1 \underset{t}{\overset{s}{\rightrightarrows}} 0 \right\}$ into the cat of sets. $\quad F : uG \longrightarrow Set$.

$$F_1 \underset{Ft}{\overset{Fs}{\rightrightarrows}} F_0$$

Object of $\underline{Gph}$ are functors

Homomorphisms of $\underline{Gph}$ are ???

$\mathcal{G}$ graph-as-functor $G : \{1 \rightrightarrows 0\} \longrightarrow Set$ $\left(G1 \overset{Gs}{\underset{Gt}{\rightrightarrows}} G0\right)$

$\Uparrow f$ $\qquad\qquad$ $\left(F1 \overset{Fs}{\underset{Ft}{\rightrightarrows}} F0\right)$

$\mathcal{F}$ graph-as-functor $F : \{1 \rightrightarrows 0\} \longrightarrow Set$ $\qquad F(0) = F_0$ etc

$\underset{\sim}{f_0} : F_0 \longrightarrow G_0$

A graph hom. is a pair (f_0, f_1)

$\underset{\sim}{f_1} : F_1 \longrightarrow G_1$

such that
$$\begin{array}{ccc} F1 & \xrightarrow{Fs} & F0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ G1 & \xrightarrow[Gs]{} & G0 \end{array} \quad \& \quad \begin{array}{ccc} F1 & \xrightarrow{Ft} & F0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ G1 & \xrightarrow[Gt]{} & G0 \end{array}$$

Given categories $\mathcal{C}$, $\mathcal{D}$.

functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$

A <u>natural transformation</u> $\alpha : F \Longrightarrow G$

consists of a family

$$\alpha_C : FC \longrightarrow GC \quad (\text{morphisms of } \mathcal{D}!)$$

one for each object of $\mathcal{C}$, with the property that

For every $u : C \longrightarrow C'$ morphism in $\mathcal{C}$,

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"'] \arrow[d, "Fu"'] & GC \arrow[d, "Gu"] \\ FC' \arrow[r, "\alpha_{C'}"'] & GC' \end{tikzcd}$$

<u>commutes</u>.

$$G\left(C \xrightarrow{u} C'\right) = GC \xrightarrow{Gu} GC'$$

$$Gu \circ \alpha_C = \alpha_C \circ Fu$$

Observe that

— There is an <u>identity</u> nat. trns.

$id_F : F \Rightarrow F$ each component of which is identity

$id_{FX} : FX \xrightarrow{id_{FX}} FX$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd}$$

$$\begin{tikzcd} FX \arrow[r, "id"] \arrow[d, "Fu"'] \arrow[dr, "Fu"] & FX \arrow[d, "Fu"] \\ FY \arrow[r, "id"'] & FY \end{tikzcd}$$

— nat. trns. compose

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha"] \arrow[rr, Rightarrow, bend left, "\beta \circ \alpha"] & G \arrow[r, Rightarrow, "\beta"] & H \end{tikzcd}$$

components for

$$(\beta \circ \alpha)_X = \begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[rr, bend right, "\beta_X \circ \alpha_X"'] & GX \arrow[r, "\beta_X"] & HX \end{tikzcd}$$

$$
\begin{tikzcd}
FX \arrow[rr, "\beta_X \circ \alpha_X"] \arrow[dr, "\alpha_X"'] \arrow[ddd, "Fu"'] & & HX \arrow[ddd, "Hu"] \\
 \arrow[dr, phantom, "\mathrm{I}"] & GX \arrow[ur, "\beta_X"'] \arrow[d, "Gu"] \arrow[dr, phantom, "\mathrm{II}"] & \\
 & GY \arrow[dr, "\beta_Y"] & \\
FY \arrow[ur, "\alpha_Y"] \arrow[rr, "\beta_Y \circ \alpha_Y"'] & & HY
\end{tikzcd}
$$

$(u : X \longrightarrow Y)$

$(\beta_Y \circ \alpha_Y) \circ Fu =$
$Hu \circ (\beta_X \circ \alpha_X)$

I commutes because α is natural

II commutes β natural

$Gu \circ \alpha_X = \alpha_Y \circ Fu$ I

$Hu \circ \beta_X = \beta_Y \circ Gu$ II

$$
\begin{aligned}
(\beta_Y \circ \alpha_Y) \circ Fu &= \beta_Y \circ (\alpha_Y \circ Fu) && \text{(associativity)} \\
&= \beta_Y \circ (Gu \circ \alpha_X) && (\mathrm{I}) \\
&= (\beta_Y \circ Gu) \circ \alpha_X && \text{(assoc)} \\
&= (Hu \circ \beta_X) \circ \alpha_X && (\mathrm{II}) \\
&= Hu \circ (\beta_X \circ \alpha_X) && \text{(assoc)}.
\end{aligned}
$$

There is a category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$

having objects $F, G, H, \ldots : \mathscr{C} \longrightarrow \mathscr{D}$

morphisms are natural trns $\alpha : F \Longrightarrow G$

$(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$ cptwise composition

$\mathrm{id} =$ cptwise identity

$$\begin{aligned}
(\gamma \circ (\beta \circ \alpha))_X &= \gamma_X \circ (\beta \circ \alpha)_X \\
&= \gamma_X \circ (\beta_X \circ \alpha_X) \\
&= (\gamma_X \circ \beta_X) \circ \alpha_X
\end{aligned}$$

$$\begin{aligned}
&= (\gamma \circ \beta)_X \circ \alpha_X \\
&= ((\gamma \circ \beta) \circ \alpha)_X
\end{aligned}$$

The category of d.graphs $\equiv$ the category of functors $\{1 \rightrightarrows 0\} \longrightarrow Set$ and natural trns

DGraph = functor

DG. hom = n. transformation

Given a group or monoid G $X : G \longrightarrow Set$ consists of a set X over which G acts; maps that preserve the action ("equivariant") are exactly natural transformations between said functors.

if $G = (\mathbb{N}, +, 0)$

$\mathrm{Dyn} = \left(X, \; f : X \circlearrowleft\right)$

(X, f) consists of a functor $(\mathbb{N}, +) \longrightarrow \mathrm{Set}$

$$\bullet \overset{1}{\circlearrowleft} \longmapsto X \overset{f}{\circlearrowleft}$$

morphism of dynamical sys

$$\begin{tikzcd}
(X, f) \arrow[loop below, "f"] & X \arrow[r, "f"] \arrow[d, "u"'] & X \arrow[d, "u"] \\
(Y, g) \arrow[loop below, "g"] & Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

$u(f(x)) = g(u(x))$

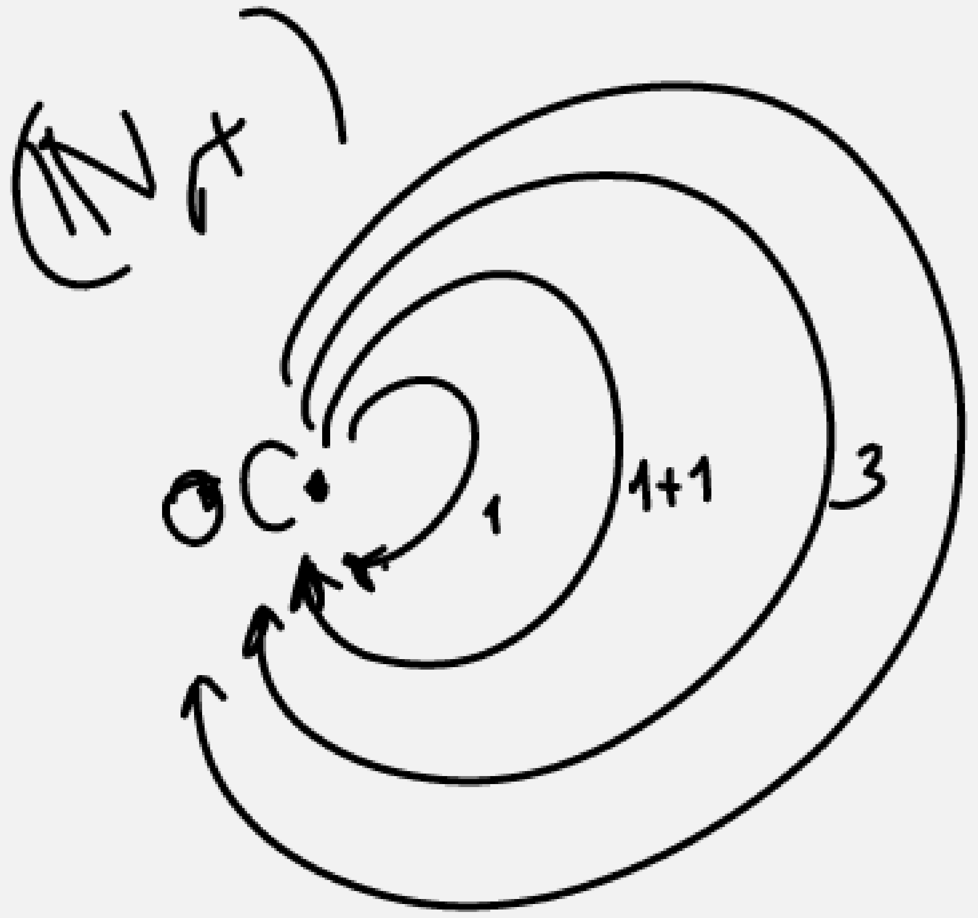
"equivariancy"

$$\begin{tikzcd}
x_0 \arrow[r, bend left, "f"] \arrow[d, mapsto, "u"'] & f(x_0) \arrow[r, bend left, "f"] \arrow[d, mapsto] & ff(x_0) \\
y_0 = u(x_0) \arrow[r, bend left, "g"] & \underline{g(u(x_0))} &
\end{tikzcd}$$

Presenting dyn sys as functors,

equivariant maps = n. transfs

$F : \underline{(\mathbb{N},+)} \longrightarrow \mathrm{Set}$ $\quad (\underset{F\bullet}{X}, \underset{F(1)}{f}) \quad f : X \longrightarrow X$

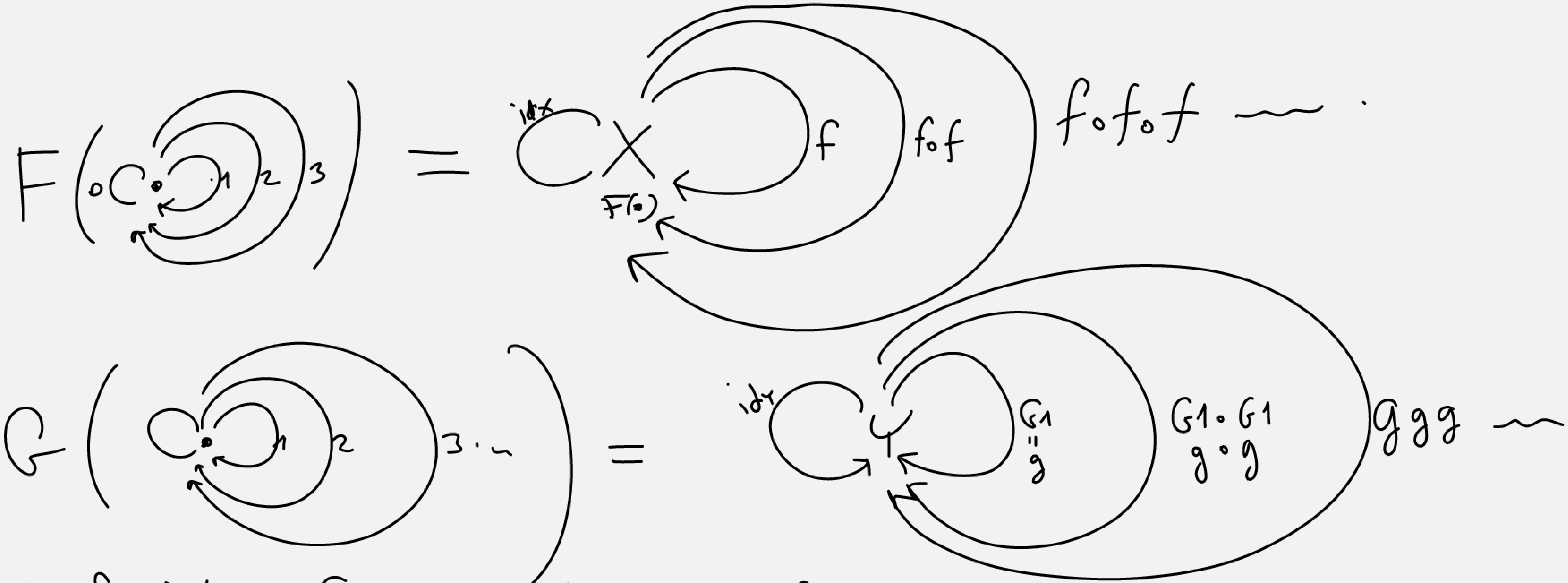


$G : (\mathbb{N},+) \longrightarrow \mathrm{Set}$ $\quad (\overset{G\bullet}{Y}, g) \quad g : Y \longrightarrow Y$

A natural trns $\alpha : F \Longrightarrow G$ consists of $\alpha_X : FX \longrightarrow GX$

Only one of them ((ℕ,+) is a category w/ one object)

$\alpha_\bullet : F\bullet \longrightarrow G\bullet$ is a function X



$$
\begin{tikzcd}
F(\bullet) = X \arrow[r, "f"] \arrow[d, "\alpha_0 = u"'] \arrow[dr, phantom, "//"] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "//"] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "//"] & X \arrow[r, "f"] \arrow[d, "u"] \arrow[dr, phantom, "//"] & X \arrow[r] \arrow[d, "u"] & \cdots \\
G(\bullet) = Y \arrow[r, "g"'] & Y \arrow[r, "g"'] & Y \arrow[r, "g"'] & Y \arrow[r] & \cdots &
\end{tikzcd}
$$

Given a set A, $\text{List}(A)$

$$= \left\{ (a_1 \,—\, a_n) \;\middle|\; \begin{array}{l} a_1 \in A \\ a_2 \in A \\ \vdots \\ a_n \in A \end{array} \quad n \in \mathbb{N} \right\}$$

$A = \{x, y\}$

$$\text{List}(A) = \{ [\,],\ [x],\ [y],\ [x,y] \neq [y,x],\ \dots \}$$

$$\begin{array}{ccc} A & \longmapsto & \text{List}(A) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \text{Set} & \longrightarrow & \text{Set} \end{array}$$

is a function

$$\begin{array}{ccc} A & (a_1, a_2) & \in \text{List}(A) \\ {\scriptstyle f}\big\downarrow & \big\downarrow{\scriptstyle \text{List}(f)} & \\ B & (f(a_1), f(a_2)) & \in \text{List}(B) \end{array}$$

The list functor has the property of being POINTED

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left=50, "\mathrm{id}"] \arrow[r, bend right=50, "\mathrm{List}"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathrm{Set} \end{tikzcd}$$

$$\eta = \underline{\lambda a.(a :: ())}$$

a

$a :: ()$

$\eta : \mathrm{id} \Longrightarrow \mathrm{List}$ has components

$$\left\{ \begin{array}{rcl} \eta_A : A & \longrightarrow & \mathrm{List}(A) \\ a & \longmapsto & [a] \end{array} \right\}$$

(btw injective functor)

natural

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathrm{List}(A) \arrow[d, "\mathrm{List}(f)"] \\ B \arrow[r, "\eta_B"'] & \mathrm{List}(B) \end{tikzcd}$$

$\mathrm{List}(f) = \mathrm{map}\ f\ [a]$

$[f(a)]$

cRing category

comm. ring

R set , $+$ addition

$\cdot$ multiplication

$(R,+,0)$ abelian group

$(R,\cdot,1)$ commutative monoid

$a\cdot(b+c) = a\cdot b + a\cdot c$

$\left((b+c)\cdot a = b\cdot a + c\cdot a\right)$

$f: R \longrightarrow S$

$f(a+b) = f(a)+f(b)$

$f(a\cdot b) = f(a)\cdot f(b)$

$f(1_R) = 1_S$

$+$ cat of Groups & homomorphisms

$$\begin{array}{ccc} \mathrm{cRing} & \xrightarrow{\cancel{GL_n}\,GL_2} & \mathrm{Grp} \\ R & \longmapsto & \cancel{GL_n}\,GL_2(R) \\ f\downarrow & & \\ S & & \end{array}$$

$GL_n(R)$ group of invertible matrices with entries in R

$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc \text{ is invertible in } R \right\}$

$$\begin{array}{ccc} \mathrm{cRing} & \xrightarrow{(-)^\times} & \mathrm{Grp} \\ R & \longmapsto & R^\times = \{ x \in R \mid \exists y \quad xy = 1 \ (\& \ yx = 1) \} \end{array}$$

claim $\det_2 : GL_2 \Longrightarrow (-)^\times$ $\quad \det_2 : GL_2(R) \longrightarrow R^\times$

$$\begin{array}{ccc} GL_2(R) & \xrightarrow{\det} & R^\times \\ GL(f)\downarrow & & \downarrow f^\times \\ GL_2(S) & \xrightarrow{\det} & S^\times \end{array}$$

$f^\times(x^{\text{unit}}) = f(x)$ (b.c. ring homom. send units to units)

$f^\times(\det(A)) = \det\left(GL(f)(A)\right)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det A = (ad - bc)$

$f(\det A) = f(ad - b \cdot c)$

$= f(a) \cdot f(d) - f(b) \cdot f(c)$

$\det \begin{pmatrix} fa & fb \\ fc & fd \end{pmatrix} = $ / same.

$$\det(A^{n\times n}) = \sum_{\sigma \in \mathrm{Sym}(n)} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$f^{\times}\big(\det(A)\big) = f(\text{big sum})$$

$$= \sum \prod f(a_{i,\sigma(i)})$$ b.c. f preserves sums & prods.

$$\det\big(\mathrm{Gl}f(A)\big) =$$ same