ITI9200 — Category theory and Applications

Exercise Sheet 1 — Of mons and men
Spring Semester

These first exercises require no category theory; they are meant to establish a theoretical minimum in three areas of mathematics that category theory constanly touches upon: order theory, vector spaces, and abstract algebra.

Exercise 1:

Find a total ordering $\trianglelefteq$ of the positive natural numbers $\mathbb{N}^\times := \{1, 2, \dots\}$ satisfying all the following properties:

- $\min(\mathbb{N}^\times, \trianglelefteq) = 1$ and $\max(\mathbb{N}^\times, \trianglelefteq) = 3$;
- $\trianglelefteq$ is *not* a well-ordering;
- $m \trianglelefteq n$ if and only if $2m \trianglelefteq 2n$, but it is not true, in general, that $m \trianglelefteq n$ if and only if $km \trianglelefteq kn$ for $k \neq 2$.

Exercise 2:

Let $\mathbb{C}$ be the field of complex numbers. Consider the operation

$$- \wedge - : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}, \quad (x, y) \mapsto x \wedge y$$

defined by

$$x \wedge y = \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$$

- Verify that this is a bilinear and antisymmetric product on vectors of $\mathbb{C}^{2n}$.
- Every $2n \times 2n$ matrix with complex entries can be regarded as a list of length $2n$ of vectors of $\mathbb{C}^{2n}$: in symbols, there is an identification $M(2n, \mathbb{C}) = (\mathbb{C}^{2n})^{2n}$. For every such matrix $A = (a_1, \ldots, a_n)$ where each $a_i \in \mathbb{C}^{2n}$, define the map $\vartheta : M(2n, \mathbb{C}) \to \mathbb{C}$ by the formula

$$\vartheta(A) = \frac{1}{2^n n!} \sum_{\sigma \in S(2n)} (-1)^{|\sigma|} \prod_{i=1}^{n} a_{\sigma i} \wedge a_{\sigma(n+i)}$$

Prove that for every $2n \times 2n$ matrix $\vartheta(A) = \det A$.

Exercise 3:

- Let a, b be elements in a ring R. If $1 - ba$ is left-invertible, show that also $1 - ab$ is left-invertible, and construct a left inverse for it explicitly. Similarly, if $1 - ba$ is invertible, show that $1 - ab$ is also invertible, and construct its inverse explicitly.
- Suppose an element a in a ring R has a right inverse b, but no left inverse. Show that then the set $S_a = \{x \mid ax = 1\}$ is infinite.