# ITI9200 — Category theory and Applications

Exercise Sheet 2 — Trimurti: Categories/Functors/Naturality
Spring Semester

Basic category theory rests on three conceptual pillars: categories, functors, and natural transformations. The exercises below are meant to make you familiar with these notions and their basic properties, and extend a little bit the perspective on the definition of category.

## Some preliminary definitions

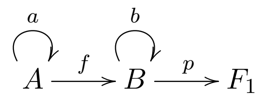
**Definition 1.1.**

1. A *partial binary algebra* is a pair $(X, *)$ consisting of a class $X$ and a partial binary operation $*$ on $X$; i.e., a binary operation defined on a subclass of $X \times X$. (The value of $*(x, y)$ is denoted by $x * y$.)
2. If $(X, *)$ is a partial binary algebra, then an element $u$ of $X$ is called a *unit* of $(X, *)$ provided that
$$x * u = x \quad \text{whenever } x * u \text{ is defined,}$$
and
$$u * y = y \quad \text{whenever } u * y \text{ is defined.}$$

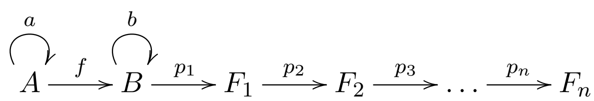
**Definition 1.2.** An *object-free category* is a partial binary algebra $\mathbf{C} = (M, \circ)$, where the members of $M$ are called *morphisms*, that satisfies the following conditions:

1. *Matching Condition:* For morphisms $f$, $g$, and $h$, the following conditions are equivalent:
   (a) $g \circ f$ and $h \circ g$ are defined,
   (b) $h \circ (g \circ f)$ is defined, and
   (c) $(h \circ g) \circ f$ is defined.
2. *Associativity Condition:* If morphisms $f$, $g$, and $h$ satisfy the matching conditions, then
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
3. *Unit Existence Condition:* For every morphism $f$ there exist units $u_C$ and $u_D$ of $(M, \circ)$ such that $u_C \circ f$ and $f \circ u_D$ are defined.
4. *Smallness Condition:* For any pair of units $(u_1, u_2)$ of $(M, \circ)$ the class
$$\hom(u_1, u_2) = \{\, f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined} \,\}$$
is a set.

**Exercise 1:**
Let $\mathcal{Q}$ be the following directed graph:




- Determine the free category $F\langle\mathcal{Q}\rangle$ on $\mathcal{Q}$ and prove that its set of morphisms determines a regular language in the alphabet $\Sigma = \{a, b, f, p\}$.
- Generalize as follows: if $\mathcal{Q}_n$ is the directed graph




  so that the previous $\mathcal{Q}$ is $\mathcal{Q}_1$, prove that the set of morphisms of $F\langle\mathcal{Q}_n\rangle$ determines a regular language in the alphabet $\Sigma = \{a, b, f, p_1, \ldots, p_n\}$. Is this still true if $n \to \infty$?

**Exercise 2:**
Define the following categories associated to the functor $S_A : X \mapsto 1 + A \times X$.

- $\nabla S$ has objects the pairs $(X, t)$ where $t \in S_A X$ is an element; a morphism $(X, t) \to (Y, v)$ in $\nabla S$ consists of a function $f : X \to Y$ such that the function $Sf$ sends $t$ to $v$:

$$Sf : 1 + A \times X \to 1 + A \times Y : t \mapsto v$$

- $\mathbf{coAlg}(S)$ has objects the pairs $(X, \xi)$ where $\xi : X \to S_A X$ is a function; a morphism $(X, \xi) \to (Y, \theta)$ in $\mathbf{coAlg}(S)$ consists of a function $f : X \to Y$ such that

$$Sf \circ \xi = \theta \circ f.$$

Recall, or learn for the first time, that

- an initial objects in a category $\mathcal{C}$ is an object $I$ such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $I \to X$;
- a terminal objects in a category $\mathcal{C}$ is an object $T$ such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $X \to T$.

Prove that $\mathbf{coAlg}(S)$ has a terminal object; prove or disprove that $\nabla S$ has an initial object.

**Exercise 3:**

A *construct* consists of a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and $U : \mathcal{C} \to \mathbf{Set}$ is a faithful functor. Two constructs $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are *(strongly) equivalent* if there exist two functors $F, G$ such that the two triangles

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathbf{Set} & \end{array} \qquad \begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ & {\scriptstyle V}\searrow \quad \swarrow{\scriptstyle U} & \\ & \mathbf{Set} & \end{array}$$

commute, and such that $F \circ G = \mathsf{id}_{\mathcal{D}}$ and $G \circ F = \mathsf{id}_{\mathcal{C}}$. When the construct functor $U$ associated to a certain category is implicitly understood (for example, when $\mathcal{C}, \mathcal{D}$ are categories of algebraic structures) we say that $\mathcal{D}, \mathcal{D}$ are *concretely equivalent*.

Are the following pairs concretely equivalent?

- $\mathcal{C} = \mathbf{Cat}$ is the category of categories and functors, defined in the usual way, while $U$ sends a category to its set of arrows; $\mathcal{D}$ is the category of object-free categories, defined above, and $V$ sends $(M, \circ)$ to the set $M$.
- the category **Kop** of *Kuratowski spaces*, defined via an interior operator, and the category **Top** of topological spaces defined via a family of closed subsets:
  - a *topological space* consists of a pair $(X, \tau)$ where $\tau \subseteq 2^X$ is a collection of subsets of $X$, such that
    1. $\varnothing, X \in \tau$;
    2. if $I$ is a set and $A_\bullet : I \to \tau$ an $I$-indexed family of elements $A_i \in \tau$, then $\bigcup_i A_i \in \tau$;
    3. if $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

    An element of $\tau$ is called an *open subset*; an element of the form $X \setminus A$ for $A \in \tau$ is called a *closed subset*.
  - A *Kuratowski space* consists of a set $X$ equipped with a monotone function

    $$j : 2^X \to 2^X$$

    called the *interior operator* of $X$, satisfying the following properties:
    1. $jX = X$;
    2. for all $S \in 2^X$, $j(S) \subseteq S$;
    3. for all $S \in 2^X$, $j(j(S)) = j(S)$;
    4. for all $S, T \in 2^X$, $j(S) \cap j(T) \subseteq j(S \cap T)$.

  Both $U$ and $V$ send a topological or Kuratowski space to its underlying set.