ITI9200 — Category theory and Applications

Exercise Sheet 4 — Adjoints
Spring Semester

Definition 0.1 (Unity and identity of opposites)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is called a *unity-and-identity-of-opposites* (UIO) if

- F has both a left and a right adjoint, $I \dashv F \dashv J$;
- I is full and faithful.

Exercise 1:

- Prove that F is a UIO if and only if
  - F has both a left and a right adjoint, $I \dashv F \dashv J$;
  - J is full and faithful.
- Find conditions under which the unique functor $! : \mathcal{A} \to \mathbf{1}$ is a UIO, if $\mathbf{1}$ is the 'point' category (a single object and its identity, $\bullet \circlearrowleft_{id}$)
- Let $\mathcal{A}$ be small, and X an object in $\mathcal{A}$; is $\mathcal{A}(X, -) : \mathcal{A} \to \mathbf{Set}$ a UIO?
- prove or disprove that the functor $O : \mathbf{Cat} \to \mathbf{Set}$ sending a (small) category to its set of objects is a UIO; is $A : \mathbf{Cat} \to \mathbf{Set}$ sending a (small) category to its set of arrows a UIO?
- prove or disprove that the functor $U : \mathbf{Gph} \to \mathbf{Set}$ sending a directed graph to its set of vertices is a UIO; if not, find sufficient conditions on a category of graphs with more structure, so that $U^+ : \mathbf{Gph}^+ \to \mathbf{Set}$ is a UIO.
- Prove or disprove: if $\mathcal{A}$ is equivalent to $\mathcal{A}'$, and $F : \mathcal{A} \to \mathcal{B}$ is a UIO, then there is a UIO $F' : \mathcal{A}' \to \mathcal{B}$ as well. Prove or disprove: if $F : \mathcal{A} \to \mathcal{B}$ is a UIO, and $G : \mathcal{B} \to \mathcal{C}$ is a UIO, then $GF : \mathcal{A} \to \mathcal{C}$ is a UIO.

$\star$! Consider the diagram

$$\mathcal{B} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{C}$$

where F, G are UIOs; figure out a way to construct a commutative diagram of UIOs

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle G}\downarrow & \searrow^{H} & \downarrow{\scriptstyle G'} \\ \mathcal{B} & \xrightarrow[F']{} & \mathcal{B} \wedge \mathcal{C} \end{array}$$