On the Fibration of Algebras

arXiv:2408.16581

j/w Ahman, Coraglia, Castelnovo, Martins-Ferreira, Reimaa

Fosco Loregian

Tallinn University of Technology

Hopf Algebras and Monoidal Categories Ferrara

Plan d'œuvre

1. What is a 'fibration of algebras'
2. A motivating example: Cartier-Gabriel-Konstant
3. Exact sequences
4. More structural results

The subject of our study will be functors

$$F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$$

- $\mathcal{A}$: the category of parameters;
- $\mathcal{X}$: the category of carriers.

Equivalently,

$$F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$$

Fibrations of algebras

To each endofunctor $F_A = F(A, -)$ one can associate

1. the category of F-algebras $Alg(F_A)$
2. the category of F-coalgebras $coAlg(F_A)$
3. the category of EM-algebras $EM(F_A)$ if F_A is a monad
4. the category of coEM-algebras $coEM(F_A)$ if F_A is a comonad
5. (Kleisli, coKleisli...)

Each of these associations defines a pseudofunctor (1. and 3. contravariant; 2. and 4. covariant)

$$\mathcal{A} \longrightarrow \mathbf{Cat}$$

as such (under the Grothendieck construction) a split fibration (1. and .3) or opfibration (2. and 4.)

Fibrations of algebras

Scope of this work:

- Study the op/fibrations

$$\begin{array}{ccccc} \mathcal{A} \ltimes^{Alg} \mathcal{X} & \mathcal{A} \ltimes^{coAlg} \mathcal{X} & \mathcal{A} \ltimes^{EM} \mathcal{X} & \mathcal{A} \ltimes^{coEM} \mathcal{X} & \mathcal{A} \ltimes^{Kl} \mathcal{A} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} & \mathcal{A} \end{array}$$

- Find examples
- Develop a general theory of these gadgets

Keywords:

representation theory, Hopf algebras, category of modules, categorical logic and type theory, graded monads, formal languages, coalgebras, ...

Fibrations of algebras

Let k be a char 0 field, algebraically closed.

> [CGK reconstruction theorem] *Every cocommutative Hopf algebra H over k arises from a semidirect product of a group G (its group algebra $k[G]$) acting on a Lie algebra L over k.*

$\mathbf{CCHopf}_k$ is a category of the form $\mathcal{A} \ltimes^{EM} \mathcal{X}$ where

- $\mathcal{A} = \mathbf{Grp}$ is the category of groups;
- $\mathcal{X} = \mathbf{Lie}_k$ is the category of Lie algebras;
- the pseudofunctor

$$\mathbf{Grp}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

  sends a group to the category $k[G]$-**Lie** of Lie algebras with an action of $k[G]$ (=Eilenberg-Moore for the monad $k[G] \otimes -$).

Fibrations of algebras

Guiding principle

A certain property of the diagram

$$\mathbf{Lie} \longleftarrow \mathbf{CCHopf}_k \longrightarrow\!\!\!\!\!\rightarrow \mathbf{Grp}$$

can likely be better understood when generalized to a property of

$$\mathcal{X} \longleftarrow \mathcal{A} \ltimes^{EM} \mathcal{X} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}.$$

Let $T : \mathcal{A} \times \mathcal{X} \to \mathcal{X}$ be a parametric monad.

Limits in $\mathcal{A} \ltimes^{EM} \mathcal{X}$ are computed in a particularly straightforward way, created by a forgetful functor

$$\langle p, V \rangle : \mathcal{A} \ltimes^{EM} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}. \qquad (\heartsuit)$$

In fact,

Theorem

The functor $\langle p, V \rangle$ is monadic.

More is true:

Theorem

$\mathcal{A} \ltimes^{EM} \mathcal{X}$ is the Eilenberg-Moore category of a monad

$$\hat{T} : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$$

fibered over the projection $\pi_{\mathcal{A}} : \mathcal{A} \times \mathcal{X} \to \mathcal{A}$. $\hat{T}(A, X) = (A, T_A X)$

Exactness

Let's study more properties of ($\heartsuit$). Let again T be a parametric monad

Assume

- that $\mathcal{X}$ has a terminal object;
- that $\mathcal{A}$ has an initial object.

$$\mathcal{X} \underset{V}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{A} \ltimes^{EM} \mathcal{X} \underset{!}{\overset{p^T}{\underset{\longleftarrow}{\overset{\twoheadrightarrow}{\bot}}}} \mathcal{A}$$

- V has a left adjoint ($\Phi X :=$ free $T_\varnothing$-algebra; in particular $\Phi X := X$ iff $T_\varnothing \cong id$)
- p^T has a right adjoint $^!(-)$ ($^!A :=$ terminal object in the fiber/A);

Theorem

Let $\mathcal{A}, \mathcal{X}$ be pointed categories; then there is an exact sequence of left adjoints

$$1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \ltimes^{EM} \mathcal{X} \twoheadrightarrow \mathcal{A} \longrightarrow 1$$

Exactness

This POV goes pretty far:

Proposition

Let $\mathcal{A}, \mathcal{X}$ be pointed categories.
There is a 2-category $\mathbf{Seq}^1(\mathcal{A}, \mathcal{X})$ of sequences of $\mathcal{A}$ by $\mathcal{X}$

$$1 \longrightarrow \mathcal{X} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{A} \longrightarrow 1.$$

where p is a fibration.

Remark

There is a subcategory $\mathbf{Ext}^1(\mathcal{A}, \mathcal{X})$ spanned by the objects of $\mathbf{Seq}^1(\mathcal{A}, \mathcal{X})$ such that $p \circ i$ is constant at the zero object.

Theorem (teaser)

$\mathbf{Ext}^1(\mathcal{A}, \mathcal{X})$ is a symmetric monoidal category.

Torsion theories on $\mathcal{A} \ltimes^{Alg} \mathcal{X}$

Theorem (Gran, Kadjo, Vercruysse)

There is a torsion theory on the category $\mathbf{CCHopf}_k$ *where*

- *torsion objects are the primitive Hopf algebras (generated by their primitive elements);*
- *torsionfree objects are group Hopf algebras (generated by their grouplike elements).*

The result in [GKV] is a particular instance of

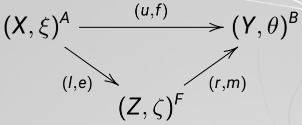
Theorem

Let $\mathcal{A}, \mathcal{X}$ be semiabelian categories each equipped with a torsion theory $\mathfrak{T} = (\mathcal{U}, \mathcal{V})$ and $\mathfrak{S} = (\mathcal{U}', \mathcal{V}')$, let $T : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$ a parametric monad such that [assumptions].
Then, there is a torsion theory $\mathfrak{T} \ltimes^{EM} \mathfrak{S}$ induced on $\mathcal{A} \ltimes^{EM} \mathcal{X}$.

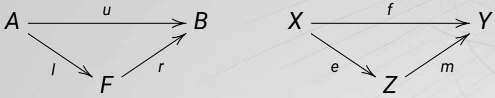
Torsion theories on $\mathcal{A} \ltimes^{Alg} \mathcal{X}$

TAL TECH

Given factorisation systems on $\mathcal{A}, \mathcal{X}$ separately, the product factorisation system on $\mathcal{A} \times \mathcal{X}$ allows to factor every morphism $(u, f) : (X, \xi)^A \to (Y, \theta)^B$ in $\mathcal{A} \ltimes^{EM} \mathcal{X}$ as follows:



where



are the factorisations in $\mathcal{A}, \mathcal{X}$ respectively.

Action of $\pi_1 B$ on fibers

Recall that any fibration of spaces $p : E \to B$ renders $E_b = p^{-1}(b)$ a $\pi_1(B)$-space; there is an analogue of this result here.

Theorem [Monads as pruned fibrations]

Let $\mathcal{A}$ have an initial object, $p : \mathcal{E} \to \mathcal{A}$ be a fibration that

- has a fully faithful left adjoint,
- admits at least coproducts of the form $\varnothing_A + E$ for every $A \in \mathcal{A}$, $E_0 \in \mathcal{E}_\varnothing$;

these are the objects of the full subcategory $\mathbf{pFib}/\mathcal{A}$ of pruned fibrations over $\mathcal{A}$.

Then there is a canonical way to build a parametric monad $T^p : \mathcal{A} \to [\mathcal{E}_\varnothing, \mathcal{E}_\varnothing]$ so that the base of p acts over the fiber of p on the initial object.

Moreover, T^p is a monad such that $T^p_\varnothing = id$. (A pruned monad) This is part of a reflection

$$\mathbf{pFib}/\mathcal{A} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathbf{pMnd}(\mathcal{A})$$

identifying pruned monads with their fibration of EM-algebras.

Conclusions

- more examples in representation theory, type theory, the categorical semantics of transition systems... (ideally, there are 'as many examples as there are endofunctors')
- more theorems about the 2-category of extensions, that mimick the 'hands-on' theory of *Ext* groups;
- consequences of monadicity: $\mathcal{A} \ltimes^{EM} \mathcal{X}$ is a 'rigid' object whose nature is specified by the formal theory of monads in $\mathit{Fib}/\mathcal{A}$
- generalizations to a theory of 1-cells of $\mathit{Fib}/\mathcal{A}$ monadic over a (bi)fibration other than the trivial one...

It's a long-term project.