Grothendieck Homotopy Theory in a nutshell

Fosco Loregian

ma per seguir virtute e canoscenza

SISSA

December 11, 2013

Nerve-realization paradigm

Let **D** be a cocomplete small category; it is a common knowledge that for any other small category **C** there exists an equivalence

$$\mathrm{Fun}(\mathbf{C},\mathbf{D}) \cong \mathrm{Adj}(\widehat{\mathbf{C}},\mathbf{D})$$

realized sending $Q\colon \mathbf{C}\to\mathbf{D}$ in the adjoint pair $|-|_Q \dashv N_Q$, where we consider the functors

- **D**-shaped nerve: $N_Q\colon \mathbf{D}\to\widehat{\mathbf{C}}\colon d\mapsto (N_Q(d)\colon c\mapsto \mathbf{C}(Q(c),d))$;
- **D**-oidal realization, $|-|_Q = \mathrm{Lan}_y Q$, obtained from the diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\;Q\;} & \mathbf{D} \\ {\scriptstyle y}\big\downarrow & \nearrow_{\mathrm{Lan}_y Q} & \\ \widehat{\mathbf{C}} & & \end{array}$$

Checking that $\mathrm{Lan}_y Q \dashv N_Q$ is mere *adjoint-yoga* and the assignment $Q\mapsto (|-|_Q\dashv N_Q)$ is functorial in Q by evident reasons; fully faithfulness of this correspondence can be proved via

$$\mathrm{Nat}(Q,Q') \cong \mathrm{Nat}(Q,\mathrm{Lan}_y Q'\circ y) \cong \mathrm{Nat}(\mathrm{Lan}_y Q,\mathrm{Lan}_y Q').$$

Localization theory in a nutshell

Idea: categorify localization of rings categories ∼ multi-object monoids).

- Everybody is comfortable with the construction building $\mathbb{Q}$ out of $\mathbb{Z}$;
- Everybody is comfortable with the construction building $R[S^{-1}]$, out of a commutative domain R: for S a multiplicative system S,

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ {\scriptstyle i}\downarrow & \nearrow {\scriptstyle \exists! h} & \\ R[S^{-1}] & & \end{array}$$

Localization theory for categories is the exact formal analogue in the world of many-object-monoids: for $S \subseteq \mathrm{Mor}(\mathbf{C})$,

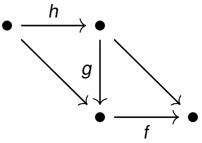
$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ {\scriptstyle i}\downarrow & \nearrow {\scriptstyle \exists! H} & \\ \mathbf{C}[S^{-1}] & & \end{array}$$

- It always exists in a suitably large universe ([Gabriel-Zisman], 1967), but it is *extremely* difficult to describe explicitly.

Homotopical and model categories

Definition

A homotopical category consists of a pair $(\mathbf{C}, \text{WK})$ where WK is a class of arrows in $\mathbf{C}$ satisfying the 2-out-of-6 property: given a commutative diagram



if both $fg, gh \in \text{WK}$, then all $f, g, h, fgh \in \text{WK}$.

Quillen's motivation to introduce model categories was to find a way to express the fact that two homotopical categories (hopefully with additional structure) give rise to the same "homotopy theory" once localized *à la* Gabriel-Zisman.

The other is that GZ-localization becomes far more easier in the presence of a model structure on $(\mathbf{C}, \text{WK})$: $\mathbf{C}[\text{WK}^{-1}] \cong \mathbf{C}_{\text{cf}}/\simeq$.

Two homotopical categories $(\mathbf{C}, \text{WK})$, $(\mathbf{D}, \text{WK}')$ are Quillen equivalent if they are linked by an adjunction

$$F\colon \mathbf{C} \leftrightarrows \mathbf{D}\colon G$$

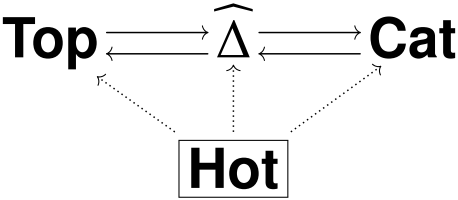
which becomes an equivalence once localized; the extremely beautiful simmetry hidden in the theory entails that even the class of homotopical categories has an homotopical structure on its own: **being Quillen equivalent is kind of like being homotopic**.

So we are allowed to think that two models (i.e. two categories equipped with a class of “weak equivalences”) describe the same homotopy theory if they are in the same “homotopy class”.

Among other things, [Quillen] shows that simplicial sets and (certain) topological spaces really are different models for the same homotopy theory: there is a Quillen equivalence (given by the nerve-realization paradigm) $\mathbf{sSet} \leftrightarrows \mathbf{Top}$.

We are interested in unraveling other (maybe all?) models for the homotopy theory of topological spaces

Aspects of Hot



The Hot category: topological model

In all what follows **Top** will denote be a nice category to do Algebraic Topology (either compactly generated Hausdorff spaces **CGHaus** or CW-complexes **CW** will do the trick).
The homotopy category of **Top** is defined to be the GZ-localization of **Top** with respect to the class of *homotopy equivalences*, namely the class $\mathcal{W}$ of arrows such that $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism (between pointed sets, groups, abelian groups) for any $n \geq 0$.

Now, the homotopy category of **Top** has many other models! We concentrate on two of them for the moment, building Quillen equivalences between **Top** and the category of simplicial sets and small categories.

The simplicial model for **Hot**

Recall that the category of simplicial sets is defined to be the category $\widehat{\Delta}$ of presheaves on the category of nonempty, finite linear orders Δ, having

- as objects finite nonempty linearly ordered sets $\{0 < \cdots < n\}$;
- as arrows, (weakly) monotone mappings betweens sets.

Being **Top** cocomplete, the nerve-realization paradigm applied to $j\colon \Delta \to \mathbf{Top}\colon [n] \mapsto \Delta^n \subset \mathbb{R}^{n+1}$ gives an adjoint pair

$$\widehat{\Delta} \underset{N_j}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{Top}$$

which coincides with the classical nerve-singular complex of simplicial sets-spaces: given $X_* \in \widehat{\Delta}$, the Kan extension $|X_*|$ can be expressed as the coend

$$|X_*| = \int^{m\in\Delta} K_m \times \Delta^m \cong \mathrm{coeq}\Big(\coprod_{m\to n} K_m \times \Delta^n \rightrightarrows \coprod_{n\in\Delta} K_n \times \Delta^n \Big).$$

The adjunction $|\cdot| \dashv N_j$ is a Quillen equivalence: we can characterize simplicial weak equivalences to be those simplicial maps whose geometric realization is a (topological) homotopy equivalence.

The categorical model for **Hot**

We can repeat the trick: the nerve-realization paradigm applied to the functor $\iota\colon \Delta \to \mathbf{Cat}$ regarding the poset $[n]$ as a category, gives an adjoint pair

$$\widehat{\Delta} \underset{N}{\overset{|\cdot|}{\rightleftarrows}} \mathbf{Cat}$$

where a simplicial set X_* is categorically realized (same definition as a Kan extension along the Yoneda embedding), and the categorical nerve is defined to be the classical nerve of a category sending $\mathbf{C}$ to the simplicial set $[n] \mapsto \mathbf{Cat}(n, \mathbf{C})$.

A categorical weak equivalence is defined to be a functor $F\colon \mathbf{C} \to D$ such that the simplicial map induced between the nerves is a simplicial weak equivalence; categorial weak equivalences form the homotopical class $\mathcal{W}_\infty \subset \mathrm{Mor}(\mathbf{Cat})$.
Again, there is an adjoint pair which realizes a Quillen equivalence between **Top** and **Cat**; anyway the nerve is not the right functor: instead one has to consider the *category of elements*

$$\int^{\Delta}(-)\colon \widehat{\Delta} \to \mathbf{Cat}$$

Definition

Let $(f^*, f_*) \colon \mathcal{E} \to \mathcal{F}$ be a geometric morphism between (Grothendieck) toposes; it is called an Artin-Mazur equivalence if for any $m \geq 0$ the morphism induced in (the) cohomology (of the toposes)

$$H^m(\mathcal{F}, P) \to H^m(\mathcal{E}, f^*P)$$

is invertible for any locally constant sheaf $P \in \mathcal{F}$.

Remark

An alternate description of categorical weak equivalences can be obtained via topos cohomology and AM-equivalences; a functor $F \in \mathbf{Cat}(\mathbf{C}, \mathbf{D})$ belongs to $\mathcal{W}_\infty$ iff the geometric morphism induced by F between presheaf toposes $\widehat{\mathbf{C}} \leftrightarrows \widehat{\mathbf{D}}$ is an AM-equivalence.

Sheaves on the space $B\mathbf{C} = |N\mathbf{C}|$ are in AM-equivalence with $\widehat{\mathbf{C}}$; we can accept as a classical result (see for example [Moerdijk]) that the cohomology of the topos of sheaves on a tame space corresponds degree-wise with the classical cohomology of X. Then a continuous map of spaces is a quasi-isomorphism iff the induced geometric morphism is an AM-equivalence.

The central problem adressed by Grothendieck in its monumental letter is: how can we determine the modelizers of the category **Hot**, namely *all* the homotopical categories $(\mathbf{C}, \mathcal{W})$ whose GZ-localization is equivalent to the category **Hot**?
In particular, Grothendieck tries to find the *canonical* modelizers of **Hot**, i.e. those modelizers such that the class of weak equivalences is uniquely determined by some other data inherent in the category **C**.

Remark (Induced homotopical structure on $\mathcal{E}$)

It's worth to mention that **AM-equivalences define an homotopical structure on any topos**: if we denote $\mathcal{E}/X$ the co-slice category on $X \in \mathcal{E}$, we can define $\mathcal{W}_{\widehat{\mathbf{C}}} = \{A \xrightarrow{\phi} A' \mid \mathcal{E}/A \to \mathcal{E}/A'$ è un'equivalenza di Artin-Mazur$\}$. In particular if $\mathcal{E} = \widehat{\mathbf{C}}$, then a morphism between presheaves $P \to Q$ is a weak equivalence iff the geometric morphism $\mathcal{E}/P \to \mathcal{E}/Q$ is an AM-equivalence.

Definition

Consider the functor

$$\imath_{\mathbf{C}} \colon \widehat{\mathbf{C}} \longrightarrow \mathbf{Cat}$$

sending a presheaf to its category of elements (via the so-called Grothendieck construction), and denote

$$\bar{\imath}_{\mathbf{C}} \colon \widehat{\mathbf{C}}[\mathcal{W}_{\widehat{\mathbf{C}}}^{-1}] \longrightarrow \mathbf{Cat}[\mathcal{W}_{\infty}^{-1}]$$

the induced functor between homotopy categories.

Notice that this situation generalizes the previous one, since if $\mathbf{C} = \Delta$ then $\imath_{\mathbf{C}} = \int^{\Delta}(-) \colon \widehat{\Delta} \to \mathbf{Cat}$, is the functor we defined before, giving a Quillen equivalence. This simple remark lead Grothendieck to define the

> ***Central problem****: find necessary and sufficient conditions so that $\bar{\imath}_{\mathbf{C}}$ is an equivalence of categories.*

Test Categories

$$\mathbf{Cat}[\mathcal{W}_{\infty}^{-1}] \cong \mathbf{C}[\mathcal{W}_{\widehat{\mathbf{C}}}^{-1}]$$

As it is stated, the central problem is extremely difficult. So Grothendieck gives an additional hypotesis leading to the extremely neat theory of *test categories*:

- The functor $i^*_{\mathbf{C}} : \mathbf{Cat} \to \widehat{\mathbf{C}}$, right adjoint to $i_{\mathbf{C}}$, sending a category $\mathbf{D}$ in the presheaf $c \mapsto \mathbf{Cat}(\mathbf{C}/c, \mathbf{D})$, is homotopical. In such a case the category $\mathbf{C}$ is called weak test category (WTC), and the functor induced by $i^*_{\mathbf{C}}$ between the localizations is an equivalence of categories, whose quasi-inverse is exactly $\bar{i}_{\mathbf{C}}$.
- Grothendieck obtains an extremely simple characterization of WTCs: $\mathbf{D}$ is a WTC if and only if $F = i^*_{\mathbf{C}}(\mathbf{D}) \in \widehat{\mathbf{C}}$ is an *aspherical* presehaf, i.e. such that the unique arrow $\int^{\mathbf{C}^{\mathrm{op}}} F \to 1$ in $\mathbf{Cat}$ is a weak equivalence.
- A WTC such that all its slices $\mathbf{C}/X$ are again WTC is called local test category (LTC); $\widehat{\mathbf{C}}$ is called elementary modelizer and $\mathbf{C}$ is called a *test category*, if $\mathbf{C}$ is both a LTC and a WTC.
  - ***Criterion****: $\mathbf{C}$ is a LTC if and only if $F = i^*_{\mathbf{C}}(\{0 \to 1\})$ is locally aspherical, namely the induced functor $\mathbf{C}/X \to \mathbf{C} \xrightarrow{F} \mathbf{Sets}$ is aspherical for any slicing $\mathbf{C}/X$.*
  - ***Criterion****: A LTC $\mathbf{C}$ is a test category if and only if the unique arrow $\mathbf{C} \to 1$ belongs to $\mathcal{W}_\infty$ (weak equivalences in $\mathbf{Cat}$). In such a case the category $\mathbf{C}$ is called aspherical.*

Digression: geometric shapes for HCT

The archetypal example of test category is of course the simplex category Δ; the formalism of TCs, together with the notion of *Reedy category*, gives a recognition principle to find good *geometric shapes for higher structures* i.e. reasonable models to formalize the notion of higher category and higher morphism.

The categories $\Delta, \{\Theta_n \mid n \geq 2\}, \Gamma, \Omega, \{\Psi_n \mid n \geq 2\}$ of simplices, Joyal Θ- or Segal Γ-spaces, dendroidal and globular sets,... are all examples of test categories, presenting (via the Yoneda embedding) categories (using Δ, categories are suitable simplicial sets), n-fold categories (using Θ_n, a n-fold category is a suitable presheaf on Θ_n), or globular sets (see [Leinster] and [Joyal]), multicategories (presheaves on Ω).

The category Ψ_n is defined to be the free category on the graph

$$n \underset{t_n}{\overset{s_n}{\leftleftarrows}} n-1 \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} n-2 \qquad \dots \underset{t_1}{\overset{s_1}{\leftleftarrows}} 0$$

modded out by the relations $s_i s_{i-1} = t_i s_{i-1}$ and $s_i t_{i-1} = t_i t_{i-1}$ for any $0 \leq i \leq n$. An n-globular set is a presheaf on Ψ_n.

Test functors

Test functors arise as a generalization of the simplicial nerve-realization pair. Suppose $\mathbf{A}$ is a WTC: then we would like to call (weak) test functor any functor $i\colon \mathbf{A} \to Cat$ such that the Yoneda extension $i^* = \mathrm{Lan}_y i\colon \widehat{\mathbf{A}} \to \mathbf{Cat}$ is a homotopical functor with respect to the implicit homotopical structures on both categories.
The definition of a QTC gives a natural example of weak test functor: $J\colon a \mapsto \mathbf{A}/a$.

Definition

A weak test functor consists of a functor $i\colon \mathbf{A} \to \mathbf{Cat}$, where the domain is a WTC, whose essential image is made by aspherical categories.

The following conditions are equivalent:

- $i\colon \mathbf{A} \to \mathbf{Cat}$ is a weak test functor;
- For any asphrical category $\mathbf{C} \in \mathbf{Cat}$, $i^*\mathbf{C}$ is an aspherical presheaf.

This criterion is also local: $\mathbf{A}$ is a LTC (i.e. $\mathbf{A}/X \to \mathbf{A} \to \mathbf{Cat}$ is a weak test functor for any $X \in \mathbf{A}$) if and only if for each aspherical $\mathbf{C} \in \mathbf{Cat}$, the presheaf $i^*\mathbf{C}$ is locally aspherical.

Elemental localizers

Grothendieck notices that the following "recognition principle" to identify categorical weak equivalences

Theorem (Quillen's *A*-theorem)

If $F\colon \mathbf{C} \to \mathbf{D}$ is a functor such that for any $d \in \mathbf{D}$ the functor $F/d\colon (\mathbf{C} \downarrow d) \to \mathbf{D}/d$ belongs to $\mathcal{W}_\infty$, then F itself belongs to $\mathcal{W}_\infty$.

has an extremely natural interpretation in terms of topos morphisms, which turns it into a natural criterion to recognize the asphericity of a map: this leads to the following definition

Definition (Elemental localizer)

We call elementa localizer any class $\mathcal{W} \subset \mathrm{Mor}(\mathbf{Cat})$ of arrows such that

- $\mathcal{W}$ defines a replete subcategory of **Cat**, satisfies the 2-out-of-3 property and is closed under retracts (we say that $\mathcal{W}$ is *quasisaturated*);
- If $\mathbf{A} \in \mathit{Cat}$ has a terminal object, then the terminal morphism $\mathbf{A} \to 1$ is in $\mathcal{W}$ (this is a property of $\mathcal{W}_\infty$: prove it!);
- Quillen's A-theorem applies to $\mathcal{W}$.

homotopy Kan extensions, smooth and proper functors

Choose your favourite homotopical category $(\mathbf{M}, \mathcal{W})$ and a small category I; give $\mathrm{Fun}(I, \mathbf{M})$ the Reedy homotopical structure: weak equivalences are the objectwise weak equivalences in $\mathbf{M}$. Denote $\mathrm{Fun}(I, \mathbf{M})_{\mathcal{W}}$ this homotopical category.

Now, any functor $G\colon I \to J$ induces an inverse image

$$G^*\colon \mathrm{Fun}(J, \mathbf{M})_{\mathcal{W}} \to \mathrm{Fun}(I, \mathbf{M})_{\mathcal{W}}$$

which is obviously homotopical with respect to the Reedy structures.

Definition

The *homotopy left (right) Kan extension* along G consist of the left (right) Quillen adjoint to the functor G^*:

$$\mathrm{Fun}(J, \mathbf{M})_{\mathcal{W}} \underset{\mathrm{holan}_G}{\overset{\mathrm{hoRan}_G}{\leftleftarrows \!\!\! \xrightarrow{G^*}}} \mathrm{Fun}(I, \mathbf{M})_{\mathcal{W}}$$

Theorem (Quillen-Thomason)

If $(\mathbf{M}, \mathcal{W})$ is *quillenizable*, namely if there is a model structure $(\text{WK}, \text{FIB} \cap \text{WK})$ "extending" the given homotopical structure, i.e. such that $\text{WK} = \mathcal{W}$, and furhtermore **M** is complete (cocomplete), then the right (left) homotopy Kan extension of any functor $\text{Fun}(G, \mathbf{M})_{\mathcal{W}}$ exists.

Choose $(\mathbf{M}, \mathcal{W}) = (\mathbf{Cat}, \mathcal{W}_\infty)$ (which is quillenizable): then the homotopy left Kan extensions of a functor $\text{Fun}(G, \mathbf{M})_{\mathcal{W}}$ can be characterized in a more hands-on way, by means of the Grothendieck construction for a functor $F\colon I \to \mathbf{Cat}$: if $\int F \xrightarrow{\Phi} I$ is its universal Grothendieck fibration, and $G\colon I \to J$ is a functor, then there exists $\int F \xrightarrow{G\circ\Phi} J$, inducing the functor $J \to \mathbf{Cat}\colon j \mapsto (\int F \downarrow j)$

The correspondence

$$\begin{aligned}\mathrm{Fun}(I, \mathbf{Cat}) &\longrightarrow \mathrm{Fun}(J, \mathbf{Cat}) \\ F &\longmapsto \Big(j \mapsto (\textstyle\int F \downarrow j)\Big)\end{aligned}$$

descends to GZ localizations, inducing a functor

$$\mathrm{Fun}(I, \mathbf{Cat})_{\mathcal{W}_\infty} \to \mathrm{Fun}(J, \mathbf{Cat})_{\mathcal{W}_\infty}.$$

This functor is easily seen to be isomorphic to hoLan_G.

Now, the existence of a homotopy left Kan extension can be translated almost *verbatim* to the case of a generic elemental localizer $\mathcal{W}$ di **Cat**.

We would like to address the dual problem, ensuring the existence of a *right* Kan extension: this is much more difficult, since it involves subtle set-theoretic issues, linked to the *accessibility* of the localizer. We address the interested audience to [Maltsiniotis]

Smooth and proper functors

Denote by $\mathbf{Hot}_{\mathcal{W}}(I)$ the GZ localization of $\mathrm{Fun}(I, \mathbf{Cat})_{\mathcal{W}}$, given an elemental localizer $(\mathbf{Cat}, \mathcal{W})$.

Given an elemental localizer $(\mathbf{Cat}, \mathcal{W})$ and a pullback square

$$\begin{array}{ccc} \mathbf{A}' & \xrightarrow{w} & \mathbf{A} \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \mathbf{B}' & \xrightarrow[v]{} & \mathbf{B} \end{array}$$

we say that u is a $\mathcal{W}$-proper functor if the canonical 2-cell

$$u'_! w^* \Longrightarrow v^* u_!$$

is invertible for any v, where $u_! : \mathbf{Hot}_{\mathcal{W}}(\mathbf{A}) \to \mathbf{Hot}_{\mathcal{W}}(\mathbf{B})$, $u'_! : \mathbf{Hot}_{\mathcal{W}}(\mathbf{A}') \to \mathbf{Hot}_{\mathcal{W}}(\mathbf{B}')$ denote the (localizzations of the) homotopy left Kan extensions of u, u'.

Smooth and proper functors

Denote by $\mathbf{Hot}_{\mathcal{W}}(I)$ the GZ localization of $\mathrm{Fun}(I, \mathbf{Cat})_{\mathcal{W}}$, given an elemental localizer $(\mathbf{Cat}, \mathcal{W})$.

Given an elemental localizer $(\mathbf{Cat}, \mathcal{W})$ and a pullback square

$$\begin{array}{ccc} \mathbf{A}' & \xrightarrow{w} & \mathbf{A} \\ {\scriptstyle u'}\downarrow & \lrcorner & \downarrow{\scriptstyle u} \\ \mathbf{B}' & \xrightarrow[v]{} & \mathbf{B} \end{array}$$

we say that v is a $\mathcal{W}$-smooth functor if the canonical 2-cell

$$w_!(u')^* \Longrightarrow u^* v_!$$

is invertible for any u, where $w_! : \mathbf{Hot}_{\mathcal{W}}(\mathbf{A}') \to \mathbf{Hot}_{\mathcal{W}}(\mathbf{A})$ and $v_! : \mathbf{Hot}_{\mathcal{W}}(\mathbf{B}') \to \mathbf{Hot}_{\mathcal{W}}(\mathbf{B})$ denote the (localizzations of the) homotopy left Kan extensions of w, v.

In Algebraic Geometry smooth and proper morphisms can be characterized to be those classes realizing the Beck-Chevalley isomorphism for the adjunction $(-)_! \dashv (-)^*$.

However, it is worth to mention that **the categorical and the geometric notion of smoothness and properness deeply differ**: in fact Grothendieck is able to show that in the categorical sense, the two notions are perfectly dual:

$$u\colon \mathbf{A} \to \mathbf{B} \text{ is proper } \iff u^{op}\colon \mathbf{A}^{op} \to \mathbf{B}^{op} \text{ is smooth.}$$

but this is by no means true on the geometric side!

Base-change along flat functors turns out to be a fundamental tool in Grothendieck homotopy theory: the two notions can be thought to be the "building blocks" of an elemental localizer $(\mathbf{Cat}, \mathcal{W})$: the result concluding [Maltsiniotis] in fact shows that (thm **3.2.45**)

Theorem

Any functor $F\colon \mathbf{C} \to D$ admits a factorization

$$\mathbf{C} \xrightarrow{W} \mathbf{K} \xrightarrow{U} \mathbf{D}$$

where W is a weak categorical equivalence, and U is both proper and smooth.

Bibliography

- Lenster, T., *Higher Operads, Higher Categories*, http://arxiv.org/pdf/math/0305049v1.pdf.
- Moerdijk, I., *Classifying Spaces and classifying topoi*, 1995.
- Moerdijk, I., Weiss, I., *Dendroidal sets*, Algebraic & Geometric Topology 7: 1441–1470.
- Grothendieck, A., *À la poursuite des champs*, manuscript.
- Cisinski, D. C., *Les préfaisceaux comme modèles des types d'homotopie*
- Maltsiniotis, G., *La théorie de l'homotopie de Grothendieck*, Astérisque, 301, pp. 1-140, (2005).